Universal Properties of Two-Dimensional Boson Droplets

H.-W. Hammer and D. T. Son

Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA

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We consider a system of N nonrelativistic bosons in two dimensions, interacting weakly via a short-range attractive potential. We show that for N large, but below some critical value, the properties of the N-boson bound state are universal. In particular, the ratio of the binding energies of $(N+1)$- and N-boson systems, B_{N+1}/B_N, approaches a finite limit, approximately 8.567, at large N. We also confirm previous results that the three-body system has exactly two bound states. We find for the ground state $B_3^{(0)} = 16.522\,688(1)B_2$ and for the excited state $B_3^{(1)} = 1.270\,409\,1(1)B_2$.

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The recent experimental progress with ultracold atomic gases has revived the interest in weakly coupled quantum liquids. The ability to control the parameters of the systems make trapped atomic gases ideal laboratories where theoretical ideas can be checked versus experiment. One of the fundamental parameters that can be varied in experiments is the dimensionality of space. Both one- and two-dimensional Bose-Einstein condensates (BEC's) of sodium atoms have been studied in atom traps [1]. A one-dimensional condensate of ^{7}Li atoms immersed in a Fermi sea of ^{6}Li atoms was observed in [2]. In Ref. [3], a two-dimensional BEC of cesium atoms was realized in a gravito-optical surface trap. A two-dimensional boson system has also been realized in hydrogen adsorbed on a helium surface [4].

In this Letter, we revisit the problem of weakly interacting bosons in two spatial dimensions (2D). While most previous theoretical studies were concerned with a Bose gas with repulsive interactions [5,6], we focus on attractive interactions. In particular, we consider a self-bound droplet of N ($\gg 1$) bosons interacting weakly via an *attractive,* short-ranged pair potential. Our goal is to exhibit universal properties pertaining to large finite systems, which are not in the thermodynamic limit.

We show that the system possesses surprising universal properties. Namely, if one denotes the size of the N-body droplet as R_N, then at large N and in the limit of zero range of the interaction potential,

$$R_{N+1}/R_N \approx 0.3417, \qquad N \gg 1. \tag{1}$$

The size of the bound state decreases exponentially with N: adding a boson into an existing N-boson droplet reduces the size of the droplet by almost a factor of 3. Correspondingly, the binding energy of N bosons B_N, increases exponentially with N:

$$B_{N+1}/B_N \approx 8.567, \qquad N \gg 1. \tag{2}$$

This implies that the energy required to remove one particle from a N-body bound state (the analog of the nucleon separation energy for nuclei) is about 88% of the total binding energy. This is in contrast to most other physical systems, where separating one particle costs much less energy than the total binding energy, provided the number of particles in the bound state is large.

To derive results independent of the details of the short-distance dynamics such as the ones quoted above, the N-body bound states need to be sufficiently shallow and hence have a size R_N large compared to all other length scales in the problem. A similar reasoning has been used in 3D with much success [7]. The breakdown of universality is determined by the next largest length scale in the problem, the natural low-energy length scale ℓ. Depending on the physical system, ℓ can be the van der Waals length $l_{\rm vdW}$, the range of the potential r_0, or some other scale. For realistic systems, Eqs. (1) and (2) are valid for large N, but below a critical value,

$$1 \ll N \ll N_{\rm crit} \approx 0.931 \ln(R_2/\ell) + \mathcal{O}(1). \tag{3}$$

At $N = N_{\rm crit}$ the size of the droplet is comparable to ℓ and universality is lost. If there is a large separation between R_2 and ℓ, then $N_{\rm crit}$ is much larger than 1 and the condition (3) can be satisfied.

Asymptotic freedom.—Our analysis relies strongly on the property of asymptotic freedom of 2D bosons with attractive interaction, so we briefly review this property here. The system under consideration is described by the nonrelativistic Hamiltonian

$$H = \int d^2\mathbf{x}\left[\frac{\hbar^2}{2m}|\nabla\psi|^2 - \frac{g}{2}(\psi^\dagger\psi)^2\right]. \tag{4}$$

The bosons interact via an attractive, short-ranged pair potential $-g\delta^2(\mathbf{x})$, with $g > 0$. This choice can be made because at low energies, the true potential cannot be distinguished from a δ-function potential. For convenience, we use the unit system $\hbar = m = 1$; the factors of $\hbar$ and m can be restored from dimensional analysis. In this unit system, g is dimensionless, and we assume $g \ll 1$.

In 2D, any attractive potential has at least one bound state. For the potential $-g\delta^2(\mathbf{x})$ with small g, there is exactly one bound state with an exponentially small bind-

ing energy,

$$B_2 \sim \Lambda^2 \exp(-4\pi/g), \tag{5}$$

where Λ is the ultraviolet momentum cutoff (which is the inverse of the range of the potential). Equation (5) can be obtained by directly solving the Schrödinger equation. However, this method is not practical for a system with more than a few particles.

Asymptotic freedom provides an alternative way to understand Eq. (5). In 2D nonrelativistic theory, the four-boson interaction $g(\psi^\dagger\psi)^2$ is marginal. The coupling runs logarithmically with the length scale R, and the running can be found by performing the standard renormalization group (RG) procedure. The RG equation reads

$$\frac{\partial g(R)}{\partial \ln R} = \frac{g^2(R)}{2\pi}. \tag{6}$$

Depending on the sign of g, we find two different behaviors. For repulsive interactions with $g < 0$, the coupling becomes weaker in the infrared. For $g > 0$, the coupling grows in the infrared, in a manner similar to the QCD coupling [8,9]. The dependence of the coupling on the length scale R is given by

$$g(R) = \left[\frac{1}{g} - \frac{1}{4\pi}\ln(\Lambda^2 R^2)\right]^{-1}, \tag{7}$$

so the coupling becomes large when R is comparable to the size of the two body bound state $B_2^{-1/2}$. This is in essence the phenomenon of dimensional transmutation: a dynamical scale is generated by the coupling constant and the cutoff scale.

It is natural, then, that B_2 is the only physical energy scale in the problem: the binding energy of three-particle, four-particle, etc., bound states are proportional to B_2. However, the N-particle binding energy B_N can be very different from B_2 if N is parametrically large. We now argue that B_N increases exponentially with N.

Stabilizing the size of the droplet.—We first try to use the variational method to estimate the size of the bound state. For a cluster of a large number of bosons, one can expect that classical field theory is applicable. We thus have to minimize the energy (4) with respect to all field configurations $\psi(\mathbf{x})$ satisfying the constraint

$$N = \int d^2\mathbf{x}\psi^\dagger\psi. \tag{8}$$

It is instructive to first minimize the energy with respect to the size of the bound state and, afterwards, overall shapes of the wave function. We use the following trial wave function:

$$\psi(\mathbf{x}) = \frac{\sqrt{N}}{R\sqrt{2\pi C}} f\left(\frac{r}{R}\right), \tag{9}$$

where $r \equiv |\mathbf{x}|$ and $f(r/R)$ is a function that describes the shape of the wave function. We assume that $f(r/R)$ is nonzero when $r/R \lesssim 1$ but becomes small when $r/R \gg 1$. Thus R is the size of the droplet. To satisfy the particle number constraint (8), we should take

$$C = \int d\rho\rho f^2(\rho). \tag{10}$$

The total energy is then obtained from Eq. (4) as

$$E(R) = \frac{A}{2C}\frac{N}{R^2} - \frac{B}{4\pi C^2}\frac{gN^2}{R^2}, \tag{11}$$

where A and B depend on the shape of the wave function,

$$A = \int d\rho\rho[f'(\rho)]^2, \qquad B = \int d\rho\rho f^4(\rho). \tag{12}$$

As one can see from Eq. (11), in 2D both the kinetic and potential energies scale as R^{-2}. This seems to prohibit a stable bound state: for $N < 2\pi AC/(Bg)$ the system expands to infinite size, while in the opposite regime, $N > 2\pi AC/(Bg)$, it shrinks to zero size.

However, the above estimate is too crude because it fails to account for the logarithmic running of the coupling g. We therefore replace g in Eq. (11) by the coupling at the length scale R: $g \to g(R)$. This procedure goes beyond the naive mean-field treatment and captures all leading logarithms (by using the RG), but it does not take into account all $1/N$ corrections. We see that this is sufficient for finding the parametric dependence of B_N on N.

Once g is replaced by $g(R)$ in Eq. (11), the energy $E(R)$ has a minimum at a finite R. Indeed, in the limit $R \to 0$ the coupling becomes weak, $g \to 0$, and $E(R)$ is dominated by the kinetic energy, which tries to make the system larger. In the opposite limit $R \to \infty$ the coupling constant becomes strong, and $E(R)$ is dominated by the negative potential energy, which favors smaller R.

To find the optimal R, we differentiate the energy (4), with g replaced by $g(R)$, with respect to R. We find

$$AC - \frac{Ng(R)}{2\pi}B + \frac{Ng^2(R)}{8\pi^2}B = 0, \tag{13}$$

where we have used Eq. (6). The solution is

$$g(R) = \frac{2\pi AC}{NB} + \mathcal{O}(N^{-2}). \tag{14}$$

The coupling is $\mathcal{O}(N^{-1})$, which implies the weak-coupling regime at large N. The $\mathcal{O}(N^{-2})$ correction is beyond the scope of the classical approximation. Using the RG running of the coupling constant (7), we find the optimal size of the droplet,

$$R_N = C_R R_2 \exp\left(-\frac{B}{AC}N\right), \tag{15}$$

where C_R is some numerical constant of order 1, which cannot be found at the current level of approximation due to the $\mathcal{O}(N^{-2})$ uncertainty in Eq. (14). However, we can

already see that the size of the droplet decreases exponentially as a function of the number of particles.

From Eq. (14) we see that the kinetic and potential energies cancel each other to leading order in $1/N$. For this reason, we can only estimate the energy to be

$$B_N = \frac{C_E}{R_N^2} = \frac{C_E}{C_R^2} B_2 \exp\left(\frac{2B}{AC} N\right) \tag{16}$$

(barring the possibility that there is a cancellation in the next-to-leading order in $1/N$), but cannot compute the overall constant C_E.

The shape of the droplet.—We now minimize the energy with respect to the shape of the wave function $f(r/R)$. Because of the exponential behavior of the energy as a function of N, the optimal shape is the one which maximizes the ratio $B/(AC)$, where A, B, and C are defined in Eqs. (10) and (12). This ratio is truly characteristic of the shape of the wave function—it is unchanged under the rescaling $f(\rho) \to \lambda_1 f(\lambda_2 \rho)$. The optimal shape of the wave function is therefore ambiguous up to this trivial rescaling.

Taking the variation of $B/(AC)$ over the $f(\rho)$, we find that $f(\rho)$ satisfies the equation

$$f''(\rho) + \frac{f'(\rho)}{\rho} - f(\rho) + f^3(\rho) = 0, \tag{17}$$

where we have performed the rescaling

$$f(\rho) \to \sqrt{\frac{B}{2C}} f\left(\sqrt{\frac{A}{C}}\rho\right). \tag{18}$$

The boundary condition on $f(\rho)$ is $f'(0) = f(\infty) = 0$. The solution can be found numerically by using, e.g., the shooting method. The solution, shown in Fig. 1, has a characteristic bell shape with $f(0) \approx 2.206$. For the shape given by the function $f(\rho)$ solving Eq. (17), $A = \frac{1}{2}B = C \approx 1.862$; therefore $B/(AC) \approx 1.074$. Equation (16) now can be written as

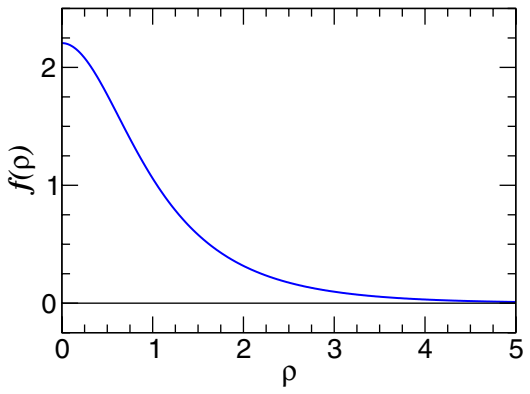

FIG. 1 (color online). Numerical solution of Eq. (17) for the boundary condition $f'(0) = f(\infty) = 0$ obtained using the shooting method.

$$B_N = c_1 B_2 c^{N-2}, \tag{19}$$

where $c \approx 8.567$, but c_1 is still unknown. Equations (1) and (2) are also recovered.

Equation (17) resembles the Hartree equation for the single-particle wave function. The above results can also be obtained in the Hartree approach, provided the running coupling is used instead of the bare one.

Three-body bound state.—We next describe our computation of the binding energies of the three-body system, which can be calculated exactly. For this purpose, we use an effective field theory and work in the Lagrangian formalism. It is convenient to introduce an auxiliary field $d \equiv \psi^2$ with the quantum numbers of two bosons (sometimes called the "dimeron") [10,11]. In terms of d and ψ, the Lagrangian density corresponding to Eq. (4) reads

$$\mathcal{L} = \psi^\dagger\left(i\partial_t + \frac{\nabla^2}{2}\right)\psi - \frac{g}{2} d^\dagger d + \frac{g}{2}(d^\dagger \psi^2 + \psi^{\dagger 2} d). \tag{20}$$

The boson propagator takes the usual nonrelativistic form $i/(p_0 - \mathbf{p}^2/2 + i\epsilon)$. It is not renormalized by interactions since all tadpole diagrams vanish in this theory. The bare dimeron propagator is simply a constant $-2i/g$. In the presence of interactions, it gets dressed by boson bubbles to all orders, leading to the full propagator:

$$i\Delta(p_0, \mathbf{p}) = -i\frac{8\pi}{g^2} \ln\left[\frac{\mathbf{p}^2/4 - p_0 - i\epsilon}{B_2}\right]^{-1}, \tag{21}$$

where the bare coupling constant g drops out of all observables in the end. The Feynman rule for the $d\psi\psi$ vertex coupling the dimeron to two bosons is ig.

The three-body binding energies are determined by the homogeneous integral equation for the three-body bound state amplitude depicted in Fig. 2. The single (double) line indicates the boson (full dimeron) propagators, while the blob is the bound state amplitude. It depends on the total energy E and the relative momentum of the boson and the dimeron. The three-body binding energies are given by those (negative) values of the total energy $E = -B_3$, for which the homogeneous integral equation shown in Fig. 2 has a nontrivial solution. The derivation of the integral equation using the Feynman rules given above proceeds as in the three-dimensional case [11]. There are only bound states if the dimeron and the third boson are in a relative S wave. The formation of bound states in the higher partial

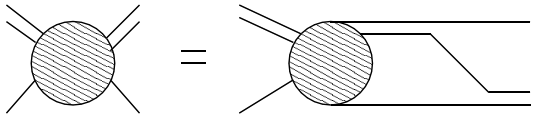

FIG. 2. The integral equation for the three-body amplitude. The single (double) line indicates the boson (full dimeron) propagator.

waves is prevented by the angular momentum barrier. Projecting onto the S wave, we obtain the equation for the bound state amplitude $F(p)$:

$$F(p) = \int_0^\infty \frac{4qdqF(q)\ln[(3q^2/4 + B_3)/B_2]^{-1}}{\sqrt{(p^2 + q^2 + B_3)^2 - p^2q^2}}. \quad (22)$$

The three-body binding energies can be obtained numerically to high precision by discretizing Eq. (22).

We find exactly two three-body bound states: the ground state with binding energy $B_3^{(0)} = 16.522\,688(1)B_2$ and one excited state with $B_3^{(1)} = 1.270\,409\,1(1)B_2$. (The numbers in parentheses give the error in the last digit.) The three-body binding energies for a zero-range potential in 2D have previously been calculated by Bruch and Tjon [12] and Nielsen *et al.* [13]. Our results are consistent with the previous calculations but more precise. Platter *et al.* have recently calculated the four-body binding energies for a zero-range potential in 2D [14]. They found exactly two bound states: the ground state with $B_4^{(0)} = 197.3(1)B_2$ and one excited state with $B_4^{(1)} = 25.5(1)B_2$.

The results for the ground state energies $B_3^{(0)}$ and $B_4^{(0)}$ are what should be compared with the asymptotic formula (2). The ratio $B_3^{(0)}/B_2 \approx 16.5$ is almost twice as large as the asymptotic value (2), while the ratio $B_3^{(0)}/B_4^{(0)} \approx 11.9$ is considerably closer. Such deviations are expected for the small values of N we are dealing with. Note, however, that the ratio of the root mean square radii of the two- and three-body wave functions is 0.306 [13], close to the asymptotic value (1).

Conclusion.—We have evaluated parametrically the bound state energy of N weakly attracting nonrelativistic bosons. Our results are obtained by minimizing the mean-field energy functional with a scale-dependent coupling. While our approximation is good enough to establish the exponential behavior of the binding energy on the number of particles, it is not sufficiently accurate for evaluating the overall coefficient in front of the exponent. It would be valuable to develop a technique capable of doing so.

We also have computed the binding energy of a system of three bosons. We have confirmed the previous finding that there are two bound states [12,13] and improved the precision of the universal values for their binding energies. One also would like to directly compute the ground state energy of the N-body system for $N > 4$ and compare the ratio $B_N^{(0)}/B_{N-1}^{(0)}$ with the asymptotic value. It would be also interesting to know the number of excited states of the N-body bound system at large N.

The many-particle bound state studied here is analogous to the nontopological soliton, or Q ball [15,16] in relativistic quantum field theory. Nonrelativistic bosons in 2D provide an interesting example where the size of the nontopological soliton is stabilized by a pure quantum effect (the running of the coupling).

An important question to explore is whether the result of this work can be extended to three-dimensional boson systems with large scattering length [17]. In contrast to the 2D case, this system displays the Efimov effect [18], and the three-body bound state energy B_3 is an independent parameter. On the other hand, the four-body bound state energy can be expressed via B_2 and B_3 [19]. One would like to know if the binding energy of N bosons can be expressed in terms of B_2 and B_3 alone and find the large N behavior of B_N and the wave function.

Finally, one should investigate the realizablity of self-bound 2D boson systems with weak interactions in experiments. According to the analysis of Ref. [14], the $1/N$ corrections to Eqs. (1) and (2) are small for $N \gtrsim 6$. Using (3), this requires $R_2/\ell \gg 600$. We are not aware of any physical system that satisfies this constraint. However, such a system could possibly be realized close to a Feshbach resonance where R_2 can be made arbitrarily large. An interesting theoretical question is the dynamics of the droplet formation [20].

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[1] A. Görlitz *et al.*, Phys. Rev. Lett. **87**, 130402 (2001).
[2] F. Schreck *et al.*, Phys. Rev. Lett. **87**, 080403 (2001).
[3] D. Rychtarik *et al.*, Phys. Rev. Lett. **92**, 173003 (2004).
[4] A. I. Safonov *et al.*, Phys. Rev. Lett. **81**, 4545 (1998).
[5] V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983).
[6] D. S. Fisher and P. C. Hohenberg, Phys. Rev. B **37**, 4936 (1988).
[7] E. Braaten and H.-W. Hammer, Phys. Rev. A **70**, 042706 (2004), and references therein.
[8] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973).
[9] H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973).
[10] D. B. Kaplan, Nucl. Phys. **B494**, 471 (1997).
[11] P. F. Bedaque, H.-W. Hammer, and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999); Nucl. Phys. **A646**, 444 (1999).
[12] L. W. Bruch and J. A. Tjon, Phys. Rev. A **19**, 425 (1979).
[13] E. Nielsen, D. V. Fedorov, and A. S. Jensen, Few-Body Syst. **27**, 15 (1999).
[14] L. Platter, H.-W. Hammer, and U.-G. Meißner, cond-mat/0405660 [Few-Body Syst. (to be published)].
[15] R. Friedberg, T. D. Lee, and A. Sirlin, Phys. Rev. D **13**, 2739 (1976).
[16] S. R. Coleman, Nucl. Phys. **B262**, 263 (1985); **B269**, 744(E) (1986).
[17] See also A. Bulgac, Phys. Rev. Lett. **89**, 050402 (2002).
[18] V. Efimov, Phys. Lett. **33B**, 563 (1970); Sov. J. Nucl. Phys. **12**, 589 (1971).
[19] L. Platter, H.-W. Hammer, and U.-G. Meißner, Phys. Rev. A **70**, 052101 (2004).
[20] C. Josserand and S. Rica, Phys. Rev. Lett. **78**, 1215 (1997).