P H Y S I C A L   R E V I E W   L E T T E R S

Quantum Calibration of Measurement Instrumentation

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By quantum calibration we name an experimental procedure apt to completely characterize an unknown measurement apparatus by comparing it with a few other calibrated apparatuses. Here we show how to achieve the calibration of an arbitrary measuring apparatus, by using it in combination with a tomographer in a correlation setup with an input bipartite system. The method is robust to imperfections of the tomographer, and works for practically any input state of the bipartite system.

The calibration of measuring apparatuses is at the basis of any experiment. Theory and experiment are unavoidably interwoven, and the calibration procedure often needs a detailed knowledge of the inner working of the apparatus, especially at extreme precisions and sensitivities, where a quantum mechanical description is needed. Here, the actual "observable" that is measured depends crucially on the microscopic details of the apparatus, the knowledge of which is needed to give a physical interpretation to the measurement.

In a quantum mechanical description, the calibration of a measuring apparatus corresponds to the knowledge of its POVM (positive operator-valued measure [1]). It gives the probability $p(n)$ of any measurement outcome n for arbitrary input state, via the Born rule

$$p(n) = \mathrm{Tr}[\rho P_n]. \tag{1}$$

In Eq. (1) ρ is the density operator of the state on the Hilbert space $\mathcal{H}$ of the system, and the POVM is given by the set of operators $\{P_n\}$ on $\mathcal{H}$. To ensure that $p(n)$ is a probability, the POVM must satisfy the positivity and normalization constraints $P_n \geq 0$, $\sum_n P_n = I$.

The concept of POVM generalizes the familiar von Neumann observable describing perfect measurements. Here the probability of obtaining the outcome n is given by $p(n) = |\langle\psi|o_n\rangle|^2$, $\{|o_n\rangle\}$ denoting a complete orthonormal basis for $\mathcal{H}$, i.e., with POVM given by the one-dimensional projectors $P_n = |o_n\rangle\langle o_n|$. The physical interpretation of the measurement is given via a quantization rule that associates a self-adjoint operator O to a classical observable, $|o_n\rangle$ being the eigenvector of O with eigenvalue o_n. However, this concept of observable does not cover many practical situations—e.g., phase estimation [2,3], joint measurements of incompatible observables [4,5], discrimination among nonorthogonal states [6,7], informationally complete measurements [8], and transmission of reference frames [9]—and here the POVM description is needed. But then, in absence of a direct physical interpretation of the measurement, we are faced with the problem of assessing the correct functioning of the measuring apparatus.

Inferring the POVM of an apparatus through the theoretical description of its functioning leads to quite involved derivations, based on different kinds of approximations. The photocounter [10] is a paradigmatic case: the number of photons claimed to be detected—usually very uncertain—is typically inferred from the cascading mechanism of the amplification process. The calibration is given essentially in terms of quantum efficiency and dark current (saturation effects generally categorize detectors into the major classes of "linear" and "single photon"). Even in a very simplified model, a theoretical description accounting for the above features is quite involved [11,12], and the resulting theoretical calibration is exceedingly indirect.

The above scenario raises the following problem: is it possible to calibrate a measuring apparatus—i.e., to determine its POVM—with a purely experimental procedure, e.g., by comparing the apparatus with few other (previously calibrated) apparatuses [13]? In this Letter we propose a method to determine a POVM experimentally. The method uses the unknown apparatus in combination with a calibrated "tomographer" on a suitably prepared bipartite system, as in Fig. 1, and the calibration results from the analysis of the correlations of the outcomes. (A tomographer is an apparatus that measures an observable tunable in a complete set called *quorum*: more details on quantum tomography will be given in the following). The basic scheme of the method stems from a previous method for the tomographic reconstruction of quantum operations [14], and generalizes a popular calibration scheme [15,16] designed to determine the quantum efficiency of a photodetector. As will be shown in the following, there is ample freedom in the choice of both the input bipartite state and the tomographer. The joint measurement must be repeated many times, analyzing the measurement outcomes with a proper tomographic algorithm [17,18]. The POVM calibration is approached in the limit of infinitely many outcomes. For a finite set of data,
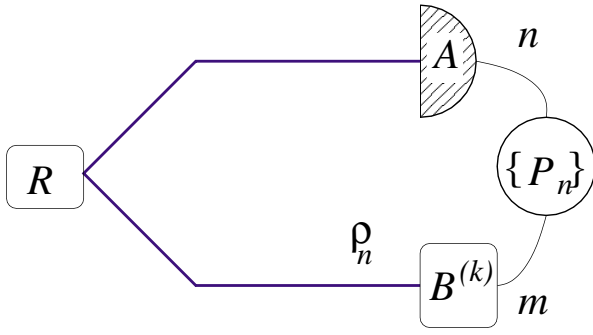
FIG. 1 (color online). Experimental setup to determine the POVM of the unknown measurement apparatus A. The apparatus A is used jointly with a tomographer on a bipartite system prepared in a predetermined state R. The tomographer measures an observable $B^{(k)}$ from the quorum $\{B^{(k)}\}$, yielding result m, whereas the unknown apparatus gives outcome n. The joint outcomes (n, m) are then processed using a tomographic algorithm, to finally obtain the POVM $\{P_n\}$ of A.

the reconstructed POVM will be affected by statistical errors, which can be precisely estimated via the tomographic algorithm. The method works for generally infinite-dimensional Hilbert space (yielding a finite number of POVM elements, corresponding to the actually occurred outcomes).

The following simple example illustrates how the procedure works. Suppose we know that the apparatus measures an observable, but we do not know which one. Denote it by the orthonormal basis $\{|o_n\rangle\}$. We can use the maximally entangled input state $|\Psi\rangle = \sum_{i=1}^d |i\rangle|i\rangle/\sqrt{d}$ in the space $\mathcal{H} \otimes \mathcal{T}$; $\mathcal{T}$ being the space of the quantum system impinging into the tomographer. The state can be equivalently written as $|\Psi\rangle = \frac{1}{\sqrt{d}} \times \sum_{j=1}^d |o_j\rangle|o_j^*\rangle$, ($|o_j^*\rangle$ denotes the vector with the complex conjugated coefficients of $|o_j\rangle$ with respect to the basis $\{|i\rangle\}$). Then, the outcome n of the unknown measuring apparatus conditions the state $\rho_n = |o_n^*\rangle\langle o_n^*|$ at the tomographer, and the POVM can be recovered using state reconstruction.

We now present the general quantum calibration procedure. Let us fix one observable at the tomographer, and denote it by $\{|b_n\rangle\}$. Upon indicating by $\{P_n\}$ the POVM of our measuring apparatus that we want to calibrate, the Born rule (1) predicts that the outcome (n, m) of the joint measurement will occur with probability

$$p(n, m) = \text{Tr}[(P_n \otimes |b_m\rangle\langle b_m|)R], \qquad (2)$$

where R is the joint state of the two quantum systems, and the POVM of the joint measurement is given by the tensor product of the individual POVM's. Upon rewriting the joint probability in terms of the conditional probability $p(m|n)$ via Bayes's rule, we conveniently introduce the

state ρ_n at the tomographer conditioned by outcome n at the unknown measuring apparatus, namely

$$p(n, m) = p(n)p(m|n) = p(n)\text{Tr}[\rho_n|b_m\rangle\langle b_m|]. \qquad (3)$$

Upon evaluating the trace in Eq. (2) in two steps, i.e.,

$$p(n, m) = \text{Tr}\left[|b_m\rangle\langle b_m|\text{Tr}_1((P_n \otimes \mathbb{1})R) \right], \qquad (4)$$

and by equating Eqs. (3) and (4) for any possible vector $|b_m\rangle$ (i.e., any possible observable), we have $\rho_n p(n) = \text{Tr}_1[(P_n \otimes \mathbb{1})R]$, namely

$$\rho_n = \frac{\text{Tr}_1[(P_n \otimes \mathbb{1})R]}{\text{Tr}[(P_n \otimes \mathbb{1})R]}, \quad p(n) = \text{Tr}[(P_n \otimes \mathbb{1})R]. \qquad (5)$$

The POVM element P_n can be recovered from the conditioned state ρ_n as follows

$$P_n = p(n)\mathcal{R}^{-1}(\rho_n), \qquad (6)$$

by inverting the map

$$\mathcal{R}(X) = \text{Tr}_1[(X \otimes \mathbb{1})R], \qquad (7)$$

where X denotes an operator on $\mathcal{H}$. The map $\mathcal{R}$ depends only on the input state R, which then must be known. Hence we need a *precalibration* stage in which we determine the joint state R (this can be done via a joint quantum tomography with two equal tomographers on the input state R). Invertibility of the map $\mathcal{R}$ corresponds to a so-called *faithful state* [14]. Since invertible maps are a dense set, then almost any quantum state R is faithful. Of course, when approaching a state corresponding to a noninvertible map, some information on the POVM $\{P_n\}$ will be lost, corresponding to increasingly large statistical errors for some matrix elements of the operators P_n (inverting a linear map is clearly equivalent to inverting an operator: for the reader who prefers operators to map, an explicit connection between operators and maps is given in Ref. [14]).

Once the inverse map $\mathcal{R}^{-1}$ has been calculated, we use quantum tomography in order to recover ρ_n. In short, quantum tomography is a method to estimate the ensemble average of an arbitrary (complex) operator X by measuring a set of observables $\{B^{(k)}\}$, called *quorum*, which span the space of operators of the system (for recent reviews on quantum tomography, see Refs. [17,19]). Typical examples of quorums are the three Pauli matrices σ_x, σ_y, and σ_z for a qubit, or the set of quadratures $X_\phi = \frac{1}{2}(a^\dagger e^{i\phi} + ae^{-i\phi})$ for a single mode of the radiation field with annihilation and creation operators a and $a^\dagger$, $\phi \in [0, \pi)$ playing the role of the observable label within the quorum $\{X_\phi\}$. (X_ϕ is measured by a homodyne detector at phase ϕ relative to the local oscillator [17].) In short, the generic operator X is expanded as $X = \sum_k \text{Tr}[XC^{(k)\dagger}]B^{(k)}$, $\{C^{(k)}\}$ denoting a dual set of the

quorum $\{B^{(k)}\}$ (the sum is replaced by an integral for continuous k), and the quantity $\text{Tr}[XC^{(k)\dagger}]$ is evaluated analytically. Notice that one can remove the effects of noise at the tomographer if the noise map $\mathcal{N}$ is invertible, by writing $X = \sum_k \text{Tr}[\mathcal{N}^{-1}(X)C^{(k)\dagger}]\mathcal{N}(B^{(k)})$.

For the tomographic reconstruction we can either: (a) average over the quorum, e.g., estimate $\langle X \rangle$ via the ensemble averages of the quorum observable as $\langle X \rangle = \sum_k \text{Tr}[XC^{(k)\dagger}]\langle B^{(k)} \rangle$ (the estimation of the density matrix element ρ_{ij} corresponding to $X = |j\rangle\langle i|$); (b) we can use the maximum likelihood approach [18]. In this case, the estimated POVM elements P_n will maximize the probability $\text{Tr}[(P_n \otimes |b_m^{(k)}\rangle\langle b_m^{(k)}|)R]$ of getting outcome n on the unknown measuring apparatus and m for the kth observable $B^{(k)}$ of the quorum, in the joint measurement $P_n \otimes |b_m^{(k)}\rangle\langle b_m^{(k)}|$ on the predetermined input state R. Equivalently, one can maximize the logarithm of this quantity and consider simultaneously all the N joint measurement outcomes, corresponding to maximizing the *likelihood functional*

$$\mathcal{L}(\{P_n\}) \equiv \sum_{i=1}^{N} \log \text{Tr}\left[(P_{n_i} \otimes |b_{m_i}^{(k_i)}\rangle\langle b_{m_i}^{(k_i)}|)R\right], \quad (8)$$

under the constraints $P_n \geq 0$ and $\sum_n P_n = I$. Other prior knowledge about P_n can be easily incorporated by adding further constraints. Moreover, we can account for a

known source of noise $\mathcal{N}$ at the tomographer, by replacing the projector $|b_{m_i}^{(k_i)}\rangle\langle b_{m_i}^{(k_i)}|$ in Eq. (8) with $\mathcal{N}(|b_{m_i}^{(k_i)}\rangle\langle b_{m_i}^{(k_i)}|)$.

The procedure to calibrate an unknown measurement apparatus can be summarized in the following steps: (i) (*precalibration*) Using two tomographers, reconstruct the input joint state R. Check whether R is faithful. (ii) (*joint measurements with the unknown apparatus*) Replace one tomographer with the unknown detector, and collect N pairs of outcomes $\{n_i, m_i\}$, $i = 1, ..., N$ in a set of joint measurements with randomly selected observable $B^{(k_i)}$ in the quorum. (iii) (*Data analysis*) From the experimental data collect the probability $p(n)$ of the outcome n at the unknown measurement apparatus, and then estimate the POVM $\{P_n\}$ using one of the above tomographic strategies—either the averaging or the maximum likelihood. In the first case evaluate the density matrix ρ_n of the state impinging in the unknown measuring apparatus, and then use Eq. (6) to recover the POVM. In the second case, evaluate the POVM directly by maximizing the likelihood functional $\mathcal{L}$ in Eq. (8) on the given set of experimental data, with the state R obtained at step (i).

In Fig. 2 we present a simulated experiment of the quantum calibration of a photocounter using homodyne tomography with the averaging strategy. The model of the calibrated detector is given in Fig. 3. Since the resulting POVM is diagonal in the photon-number basis, we limit the reconstruction to the diagonal elements only. As input state R we use a twin beam state from parametric down-conversion of vacuum, of the form $\propto \sum_m \xi^m |m\rangle \otimes |m\rangle$, where ξ is related to the amplification gain and $|m\rangle$ denotes the eigenstate of the photon number. One can easily check that the twin beam is faithful for all $\xi \neq 0$. As a typical imperfection of the tomographer, we consider nonunit quantum efficiency η_h for the homodyne detector (the noise map can be inverted as long as $\eta_h > \frac{1}{2}$ [17]). Since we reconstruct only the diagonal part of the POVM, one can easily show that there is no need to know the homodyne phase ϕ, which, however, must be randomly distributed (the knowledge of ϕ would allow to recover also the null off-diagonal elements of the POVM).
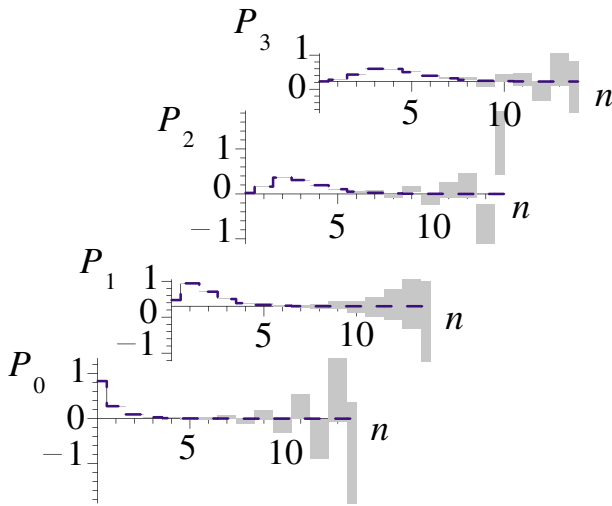

FIG. 2 (color online). Calibration of the photodetector in Fig. 3 with $\eta_p = 80\%$ and $\nu = 1$, using a twin beam with $\xi = 0.88$ (see text), and homodyne tomography with quantum efficiency $\eta_h = 90\%$. The plots are the reconstruction of the diagonal matrix elements $\langle n|P_k|n\rangle$ of the detector POVM, from a set of 5×10^6 computer-simulated data, using the averaging strategy. The reconstructed POVM is at the middle of the error bars, whereas the theoretical values, for comparison, are given by the dashed lines.
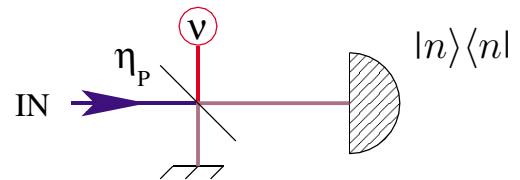

FIG. 3 (color online). Model of the photodetector calibrated in the simulations of Figs. 2 and 4. Nonunit quantum efficiency η_p and dark current with mean photon number ν are equivalent to preceding an ideal detector by a beam splitter of transmissivity η_p mixing the input signal with a thermal mode with ν average photons.
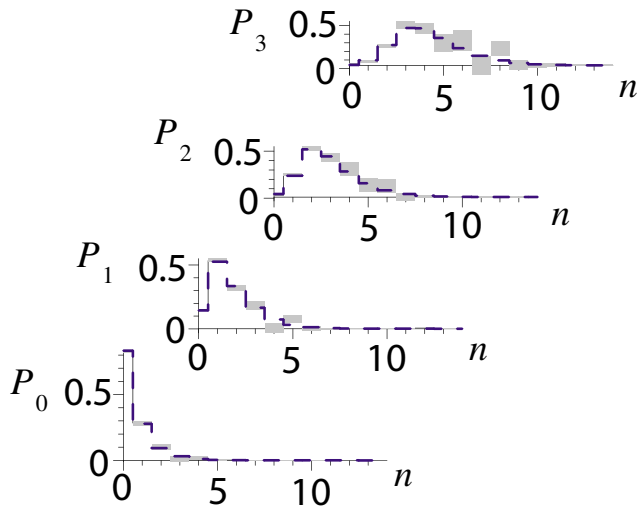
FIG. 4 (color online).   The same as in Fig. 2, but using the maximum likelihood method. Here only 5×10^4 simulated data are used. The error bars are obtained by standard boot-strapping techniques over a virtual repetition of 50 experiments. Notice how the result is statistically less noisy than that in Fig. 2, even for a 10^{-2} smaller set of data.

In Fig. 4 we present the same calibration, but using the maximum likelihood strategy. The convergence of the maximum-search algorithm is assured by the strict convexity of the likelihood functional $\mathcal{L}$ over the space of diagonal POVM's (the convergence speed, however, can be very slow in practice). In the simulation we used a blend of sequential quadratic programming routines (to perform the constrained maximization) along with expectation-maximization techniques [18]. By comparing Figs. 2 and 4 we can see how the maximum likelihood estimation is more statistically efficient (i.e., fewer data are needed to achieve the same statistical error) than the averaging strategy [20], and, in addition, the maximization of the likelihood recovers all the POVM elements simultaneously. On the other hand, compared to the averaging strategy, the maximum likelihood approach has the drawback of being biased, since one needs to put a cutoff to the Hilbert space dimension of the tomographic reconstruction and/or to the cardinality of the POVM. Both simulated experiments use realistic parameters and are feasible in the lab with current technology (see, for example, Refs. [21–24]). The major challenge of a real experiment remains the matching of modes between photocounter and homodyne detector, also ensuring that the detected modes are the same of the precalibration stage.

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