Consistency of Nonlinear System Response to Complex Drive Signals

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The consistency of a nonlinear system's response to a repeated complex waveform drive signal is an important consideration in classical and quantum systems as diverse as lasers, neuronal networks, and manufacturing plants. We show from a consideration of different characteristic waveforms that there is typically an optimal drive amplitude for the most consistent response; internal noise sources dominate for small amplitude driving while deterministic system nonlinearity reduces consistency for large amplitudes. We test this general concept and its measurement experimentally and numerically on the specific example of a laser system.

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Many nonlinear dynamical systems have an ability to generate consistent outputs when driven by a repeated external signal. *Consistency* is defined as the reproducibility of response waveforms in a nonlinear dynamical system driven repeatedly by a signal, starting from different initial conditions of the system. Consistency of dynamics is essential for information transmission in biological and physiological systems and for reproduction of spatiotemporal patterns in nature. Consistency tests could be applied in noninvasive diagnostic procedures to detect changes in system parameters due to aging, catastrophic events, or other system changes.

Several phenomena related to consistency have been studied in various nonlinear dynamical systems. Generalized synchronization has been observed [1–3] in which there is a functional relation between the dynamics of a drive and response system, but the dynamics may differ greatly in character. If one now couples two independent but identical response systems with the drive system under generalized synchronization, the response systems, starting from different initial conditions, display identical synchronization after transients have disappeared. Noise-induced synchronization is a phenomenon where two identical nonlinear systems driven by a common noise signal can be identically synchronized to each other [4]. Reliability of spike timing in neurons has been investigated [5], where neurons that are repeatedly driven by a random drive signal can fire a consistent spike train with high temporal precision. The characteristics of these three examples of drive-response systems are contained within the more general concept of consistency. Reproducibility of the response outputs with respect to a repeated drive signal is essential for all three phenomena.

In stochastic resonance [6] and coherence resonance [7], it is the periodicity or coherence of the system response that is of interest, and this is modified by an external noise signal. Here, the focus is on the reproducibility or consistency of the system response to a repeated, complex waveform drive signal. The response signal may or may not have a functional relationship to the drive signal; its consistency is a measure of the ability of the external drive to interact with and excite the system degrees of freedom in a reproducible fashion. If the drive signal is too weak, it cannot overcome the effect of internal noise sources; if it is too strong, it may deterministically destabilize the response of the nonlinearly coupled degrees of freedom.

The concept of consistency is illustrated in Fig. 1. Any complex waveform such as deterministic chaos or stochastic noise can be used as a drive signal. This drive signal is sent repeatedly to a nonlinear dynamical system (called the response system) starting from arbitrary initial conditions. Complex temporal waveforms of the response system are obtained at each repetition of the drive signal. Consistency can be defined as the ability of a system to produce identical response outputs after some transient period, when the system is driven by a repeated drive signal. In this Letter we introduce a quantitative measure of consistency, and experimentally and numerically demonstrate its measurement in the dynamics of a physical laser system driven repeatedly by a complex waveform. We observe and explain three regimes of response—a growth in consistency as the drive signal amplitude increases, followed by optimal consistency, and then a decrease of consistency as the drive amplitude increases still further. We believe that these are general features of the response of nonlinear systems with several

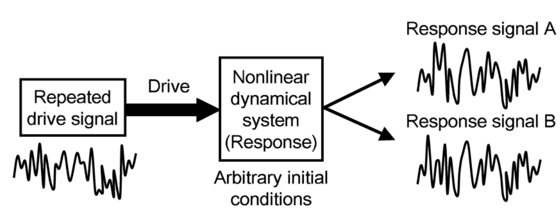

FIG. 1. Concept of consistency.

coupled degrees of freedom when sources of internal noise are present.

To test consistency, we used chaos and noise waveforms as drive signals. Our nonlinear dynamical system was a laser-diode-pumped neodymium-doped yttrium aluminum garnet (Nd:YAG) microchip laser, similar to that used for the observation of generalized synchronization of chaos [3], except no feedback loop was used. Two longitudinal modes are observed in the output of the laser in a wide range of pump power. An acousto-optic modulator (AOM) is inserted in the laser cavity to modulate the loss of the laser cavity. The digitized drive signal stored in a computer is sent to the AOM in the laser system through an amplifier and a low-pass filter to smooth the signal by using an arbitrary function generator connected with the computer. The response laser system is driven repeatedly by the same drive signal, and the temporal waveform of the response laser output is detected by using a digital oscilloscope and photodiode. We compare the detected signals for different repetitions to observe the consistency of the response waveforms.

First we used a chaotic signal generated by the same laser system with closed feedback loop as a drive signal [3]. The chaotic signal is sent to the response laser repeatedly. Temporal waveforms of the chaotic drive signal and two response laser outputs obtained from our experiment are shown in Fig. 2(a). Two consistent response outputs are clearly observed after a transient of $\sim$1 ms in Fig. 2(a), even though the drive and response signals are totally different. We next used a colored noise signal generated by a numerical algorithm [8] as a drive signal

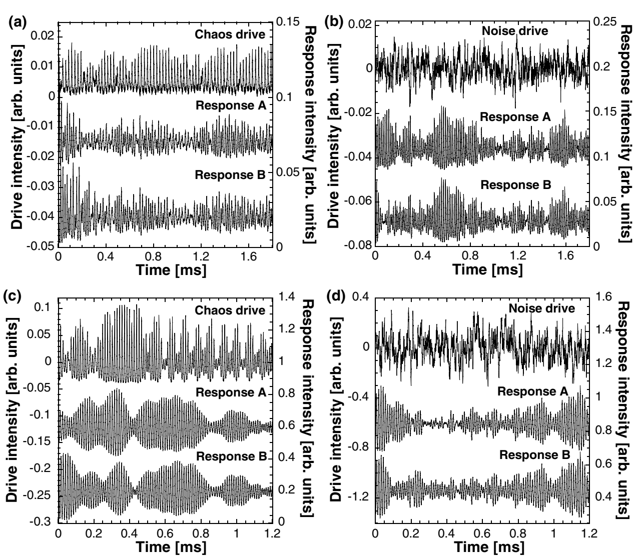

FIG. 2. Experimental results of temporal waveforms of (a) the chaotic drive signal and (b) the colored noise drive signal, and two corresponding response laser outputs. Numerical results of the temporal waveforms of (c) the chaotic drive signal and (d) the colored noise drive signal, and two corresponding response laser outputs. Consistent outputs are observed for all the figures after transient.

and sent the noise signal to the laser system repeatedly. The inverse of the correlation time for the colored noise is set to be 40 kHz. Temporal waveforms obtained from our experiments are shown in Fig. 2(b). Consistent outputs of the response laser driven by the same colored noise signal are obtained after transients as well.

Corresponding numerical results for chaotic and colored noise drive signals are shown in Figs. 2(c) and 2(d), respectively. Two response outputs starting from different initial conditions converge to the consistent outputs after transients in both cases. These numerical results agree well with our experimental observation shown in Figs. 2(a) and 2(b). We observed two longitudinal laser modes in our experiments, resulting in a dynamical system with 8 degrees of freedom. The numerical calculations to explain our experimental observations and to investigate the detailed characteristics of consistency were performed on the Tang-Statz-deMars (TSD) model which includes the effect of spatial hole burning in a two-mode laser [9]. The equations for the response laser are as follows [3]:

$$\frac{dn_0}{dt} = w_0 - n_0 - \gamma_1\left(n_0 - \frac{n_1}{2}\right)|e_1|^2 - \gamma_2\left(n_0 - \frac{n_2}{2}\right)|e_2|^2, \tag{1}$$

$$\frac{dn_1}{dt} = n_0\gamma_1|e_1|^2 - n_1(1 + \gamma_1|e_1|^2 + \gamma_2|e_2|^2), \tag{2}$$

$$\frac{dn_2}{dt} = n_0\gamma_2|e_2|^2 - n_2(1 + \gamma_1|e_1|^2 + \gamma_2|e_2|^2), \tag{3}$$

$$\frac{de_1}{dt} = \frac{K}{2}\left[\gamma_1\left(n_0 - \frac{n_1}{2}\right) - 1 - a\sin^2(z)\right]e_1 + \xi_1, \tag{4}$$

$$\frac{de_2}{dt} = \frac{K}{2}\left[\gamma_2\left(n_0 - \frac{n_2}{2}\right) - 1 - a\sin^2(z)\right]e_2 + \xi_2, \tag{5}$$

$$\frac{dz}{dt} = -\beta[z - B - RI_{\mathrm{d}}(t)], \tag{6}$$

where n_0 is the space-averaged component of population inversion density with spatial hole burning, normalized by the threshold value. n_1 and n_2 are the spatial Fourier components of population inversion density for the two longitudinal modes. e_1 and e_2 are the normalized complex electrical fields for the two-mode laser lights. $w_0 = 1.57$ is the optical pump parameter scaled to the laser threshold. $\gamma_1 = 1.00$ and $\gamma_2 = 0.93$ are the gain coefficients for the two modes. $K = \tau_{\mathrm{f}}/\tau_{\mathrm{p}} = 2.42 \times 10^4$, where $\tau_{\mathrm{f}} = 0.23$ ms is the upper state lifetime and $\tau_{\mathrm{p}} = 9.5$ ns is the photon lifetime in the laser cavity. $\beta = 1.44 \times 10^2$ is the cutoff frequency of the low-pass filter (corresponding to 100 kHz) [3]. The bias voltage of the AOM is represented by the normalized variable $B = 2.67$. $a = 4.0 \times 10^{-3}$ is the amplitude of the loss modulation, and the amplitude of the photodiode response is $R = 0.73$.

$I_d(t)$ is the drive signal. The total intensity of the response waveform is calculated from $I_r(t) = |e_1|^2 + |e_2|^2$. ξ_1 and ξ_2 are the internal white Gaussian noise due to quantum fluctuation for the two-mode laser. Time for the equations is scaled by τ_f. Numerical results are calculated with the fourth-order Runge-Kutta-Gill method.

We used chaos and colored noise waveforms as drive signals $I_d(t)$ exactly as in our experiments. For a chaotic drive signal we make a copy of the set of response equations in order to construct a system of equations simulating a chaotic drive laser with optoelectronic self-feedback [3]. We generated colored noise from the numerical algorithm [8], such that $\langle I_d(t) I_d(t+\tau)\rangle = (2D/\tau_c)\exp(-\tau/\tau_c)$, with $1/\tau_c = 40$ kHz.

To investigate the characteristics of consistency, we quantitatively define the consistency parameter C as the cross correlation of two temporal response waveforms normalized by the product of their standard deviations; i.e.,

$$C = \frac{\langle (I_A - \bar{I}_A)(I_B - \bar{I}_B)\rangle}{\sigma_A \sigma_B}, \qquad (7)$$

where I_A and I_B are the total intensities of the two response waveforms, $\bar{I}_A$ and $\bar{I}_B$ are the mean values of the two response waveforms, and σ_A and σ_B are the standard deviations of the two response waveforms. The angle brackets denote time averaging.

We calculated the time evolution of the consistency C from the time series shown in Figs. 2(a) and 2(b). The consistency is averaged over 0.1 ms and calculated continuously. The amplitude of the drive waveform is measured by its standard deviation, σ. The results are shown in Fig. 3. For the curves indicated by the label "consistent," the consistency parameter gradually increases during the transient region over the first $\sim$1 ms and stays at a value close to 1 after that, for both chaos and colored noise drive waveforms. However, when the amplitude of the drive waveforms is increased, the consistency fluctuates between -1 and 1 as shown in the curves indicated by the label "inconsistent." The achievement of consistency is dependent on the amplitude of the drive signal.

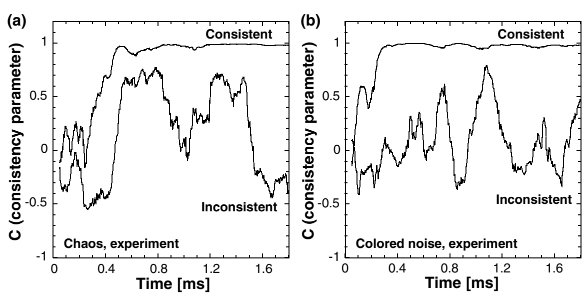

FIG. 3. Time evolution of the consistency parameter C calculated from Eq. (7). The curves indicated as consistent are obtained from the temporal waveforms shown in Figs. 2(a) and 2(b). (a) Experimental result with chaos drive signal; consistent, $\sigma = 0.035$; inconsistent, $\sigma = 0.12$; (b) experimental result with colored noise signal; consistent, $\sigma = 0.045$; inconsistent, $\sigma = 0.15$.

We varied the amplitude of the drive waveform to measure the characteristics of consistency. We computed the average of the consistency parameter C over 13 ms after the transient period. Our experimental results for the consistency as a function of the amplitude of the drive waveform is shown in Fig. 4(a) for chaos and colored noise drives. There is a maximum of the consistency curve for both drive signals as the amplitude is increased. At the very small amplitude region ($\sigma < 0.02$), consistent outputs are not observed because the internal noise-driven relaxation oscillations are dominant in the laser output. As the amplitude is increased, the drive signal overcomes the internal noise and optimal, consistent outputs are observed. When the amplitude of the drive signal is increased further, the consistency decreases and inconsistent outputs are observed. Our numerical results indicate a similar curve with the maximum peak of consistency as shown in Fig. 4(b) (solid curve) for the chaos drive signal. In order to explain the development of inconsistency in the case of strong driving, the maximal conditional Lyapunov exponent along the trajectory of the response laser output is also estimated without the

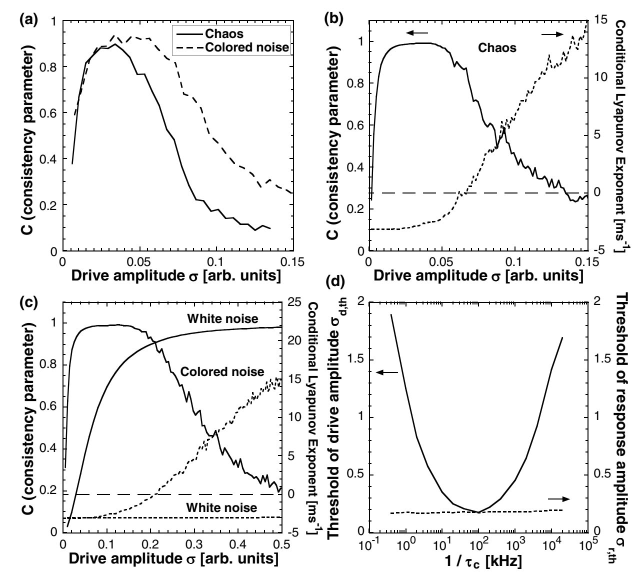

FIG. 4. (a) Experimental result of the consistency parameter C as a function of the amplitude (measured as the standard deviation) of the chaos (solid curve) and colored noise (dashed curve) drive signals. (b) Numerical result of C (solid curve) and the maximal conditional Lyapunov exponent (dotted curve) as a function of the amplitude of the chaos drive signal. (c) Numerical result of C (solid curve) and the maximal conditional Lyapunov exponent (dotted curve) as a function of the amplitude of colored noise and white noise drive signals. (d) Thresholds for inconsistency with drive amplitude (solid curve) and with response amplitude (dotted line) as a function of the inverse of the correlation time $1/\tau_c$ for the colored noise drive signal. Note that the minimum of the solid curve coincides with the laser relaxation oscillation frequency.

internal noise terms ξ [dotted curve of Fig. 4(b)] [1,3]. We found that inconsistency appears at large drive amplitudes when the sign of the conditional Lyapunov exponent changes from negative to positive. This result suggests that dynamical response of the nonlinear system generates inconsistency for large amplitude drive waveforms. On the other hand, inconsistency appears at small drive amplitudes even though the conditional Lyapunov exponent is negative. In this region, internal noise dominates and prevents consistency. Consistency is thus optimized between these two regimes for some intermediate amplitude of drive waveform.

We define the threshold for *inconsistency* as the drive amplitude at which the sign of the conditional Lyapunov exponent changes from negative to positive values. The threshold is dependent on the type of drive waveform. Our numerical results show that the threshold for the chaos drive signal ($\sigma_{\mathrm{th,chaos}} = 0.061$) is smaller than that for the colored noise drive signal ($\sigma_{\mathrm{th,color}} = 0.20$) shown as seen in Figs. 4(b) and 4(c). Since the chaotic signal is generated from the same laser system, it excites the dynamics of the laser system more effectively than the colored noise signal. This chaotic excitation thus leads more quickly to instability and reduces the threshold for inconsistency with the chaotic drive signal than with the colored noise drive. In the limit of $\tau_c \to 0$, which represents white noise, we do not find a threshold for inconsistency as shown in Fig. 4(c). This point is further clarified when we change the correlation time τ_c for the colored noise systematically. The threshold for inconsistency for the colored noise is plotted as a function of $1/\tau_c$ as shown in Fig. 4(d) (solid curve). The threshold value changes as $1/\tau_c$ is changed. The curve of the threshold for inconsistency displays a clear minimum around a frequency of 80 kHz which corresponds to the relaxation oscillation frequency of the laser. As $1/\tau_c$ differs further from the relaxation oscillation frequency, the inconsistency threshold value becomes larger. In the high bandwidth (white noise) limit, the inconsistency threshold appears to become infinite and consistency is always maintained with strong driving.

We also measure the amplitude of the *response* laser outputs at the threshold for inconsistency in Fig. 4(d) (dotted line). It is worth noting that the amplitude of the response outputs is almost constant when $1/\tau_c$ of the colored noise is changed as shown in Fig. 4(d). Therefore the threshold for inconsistency is dominated by the amplitude of the response laser output, $\sigma_{\mathrm{th,response}} = 0.18$, i.e., by the nature of its deterministic dynamics in response to the drive signal.

In conclusion, we have quantitatively defined a measure of consistency for the response of nonlinear systems to external drive waveforms. We have experimentally and numerically determined the consistency of response output in a laser driven repeatedly by different complex waveforms—chaos and colored noise with different bandwidths. An increase of consistency is observed as the amplitude of the drive signal is increased for both chaos and colored noise drive signals. After reaching an optimal value which is dependent on the internal noise strength, the consistency begins to decrease due to the deterministic dynamics of the nonlinear response as shown by the change in sign of the conditional Lyapunov exponent. These aspects of consistency of response to drive waveforms may be general features that can be observed in many driven nonlinear classical and quantum systems.

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