

Inequalities for Light Nuclei in the Wigner Symmetry Limit

Jiunn-Wei Chen,¹ Dean Lee,² and Thomas Schaefer³

¹*Department of Physics and National Center for Theoretical Sciences at Taipei, National Taiwan University, Taipei 10617, Taiwan*

²*Department of Physics, North Carolina State University, Raleigh, NC 27695, USA*

³*Department of Physics, North Carolina State University, Raleigh, NC 27695, USA
and RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA*

(Received 24 August 2004; published 7 December 2004)

Using effective field theory we derive inequalities for light nuclei in the Wigner symmetry limit. This is the limit where isospin and spin degrees of freedom can be interchanged. We prove that the energy of any three-nucleon state is bounded below by the average energy of the lowest two-nucleon and four-nucleon states. We show how this is modified by lowest-order terms breaking Wigner symmetry and prove general energy convexity results for $SU(N)$. We also discuss the inclusion of Wigner-symmetric three- and four-nucleon force terms.

DOI: 10.1103/PhysRevLett.93.242302

PACS numbers: 13.75.Cs, 21.30.-x, 21.65.+f

Weinberg was the first to apply effective field theory to the nucleon-nucleon interaction [1]. Since then, effective field theory methods for nuclear physics have been further developed and applied successfully to the two- and three-nucleon system [2]. Recently, though, there has also been progress in applying effective field theory to many-body nuclear physics using lattice methods [3] as well as other methods such as density functional theory [4].

The traditional approach to many-body nuclear physics is based on two- and three-body potential models. Although this approach has been very successful in describing the ground-state properties of light nuclei and neutron drops, the effective field theory approach offers several important advantages. First of all, effective field theory provides a direct connection with quantum chromodynamics (QCD). It explains the form of the interactions and how the strength of the interaction changes with the cutoff scale. Also, the calculations can be systematically improved and one can estimate the errors due to contributions that have been neglected. Furthermore, it has been discovered that the effective Lagrangian is often amenable to analysis so that nonperturbative results can be proven without even any numerical simulations. Effective field theory was used in [5] to prove inequalities for the correlations of two-nucleon operators in symmetric nuclear matter. In [6] upper bounds were proven for pressure in isospin-asymmetric nuclear matter and neutron matter in a magnetic field.

The idea of proving rigorous inequalities from Euclidean functional integrals is not new. In fact, the connection between effective field theory inequalities and nuclear lattice simulations is much the same as the connection between QCD inequalities [7] and lattice QCD. In this Letter we first confirm the results of [8,9] by showing that the physics of low-energy symmetric nuclear matter is close to the Wigner limit, where the isospin and spin degrees of freedom can be interchanged. In this limit $SU(2) \times SU(2)$ spin-isospin symmetry is

elevated to an $SU(4)$ symmetry. We prove that the energy of any three-nucleon state is bounded below by the average energy of the lowest two-nucleon and lowest four-nucleon states. We show how this is modified by lowest-order terms breaking Wigner symmetry and prove general energy convexity results for $SU(N)$. We also discuss the inclusion of Wigner-symmetric three- and four-nucleon force terms. Much of the physics we describe is universal and appears in other systems such as trapped Fermi gases near a Feshbach resonance [10]. Our general result for $SU(N)$ will be interesting if and when one is able to trap four or more degenerate fermionic states.

Let N represent the nucleon fields. We use $\vec{\tau}$ to represent Pauli matrices acting in isospin space and $\vec{\sigma}$ to represent Pauli matrices acting in spin space. We assume exact isospin symmetry. In the nonrelativistic limit and below the threshold for pion production, we can write the lowest-order terms in the effective Lagrangian as

$$\begin{aligned} \mathcal{L} = & \bar{N} \left[i\partial_0 + \frac{\vec{\nabla}^2}{2m_N} - (m_N^0 - \mu) \right] N - \frac{1}{2} C_S \bar{N} N \bar{N} N \\ & - \frac{1}{2} C_{\text{odd}} [\bar{N} \vec{\sigma} N \cdot \bar{N} \vec{\sigma} N - \bar{N} \vec{\tau} N \cdot \bar{N} \vec{\tau} N]. \end{aligned} \quad (1)$$

We have neglected three-nucleon terms for now but will consider them later. We have written the Lagrangian so that the operator multiplying C_{odd} flips sign under the exchange of isospin and spin degrees of freedom.

We now determine C_S and C_{odd} on a spatial lattice for various lattice spacings a_{lattice} by matching with experimental values for the nucleon-nucleon scattering lengths. We sum all nucleon-nucleon scattering bubble diagrams on the lattice, locate the pole in the scattering amplitude, and compare with Lüscher's formula relating scattering lengths and energy levels in a finite periodic box [11,12]. In Table I we have tabulated these coefficients. As noted in [13], deeply bound Efimov states will begin to appear if we make a_{lattice}^{-1} too large. In reality this is not much of a

TABLE I. Contact potential coefficients.

| a_{lattice}^{-1} (MeV) | C_S (MeV $^{-2}$) | C_{odd} (MeV $^{-2}$) |
|---------------------------------|------------------------|---------------------------------|
| 20 | -3.40×10^{-4} | -3.8×10^{-5} |
| 40 | -1.20×10^{-4} | -6.0×10^{-6} |
| 60 | -7.70×10^{-5} | -2.4×10^{-6} |
| 80 | -5.60×10^{-5} | -1.3×10^{-6} |
| 100 | -4.40×10^{-5} | -8.0×10^{-6} |

restriction since we should not let a_{lattice}^{-1} exceed the chiral symmetry breaking scale (or even m_π for the pionless theory).

We see that C_{odd} is much smaller in magnitude than C_S . In the limit $C_{\text{odd}} \rightarrow 0$, the $SU(2) \times SU(2)$ spin-isospin symmetry is elevated to an $SU(4)$ symmetry. This symmetry was first studied by Wigner [14], and arises naturally in the limit of large number of colors, N_c [15,16]. Although the 1S_0 and 3S_1 scattering lengths are quite different, the fact that both scattering lengths are large suggests we are close to the Wigner limit [8,9].

When $C_{\text{odd}} = 0$ the grand canonical partition function is given by

$$Z_G = \int DND\bar{N} \exp(-S_E) = \int DND\bar{N} \exp\left(\int d^4x \mathcal{L}_E\right), \quad (2)$$

where

$$\mathcal{L}_E = -\bar{N} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + (m_N^0 - \mu) \right] N - \frac{1}{2} C_S \bar{N} N \bar{N} N. \quad (3)$$

Using Hubbard-Stratonovich transformations, we can rewrite Z_G as

$$Z_G \propto \int DND\bar{N} df \exp\left(\int d^4x \mathcal{L}_E^f\right), \quad (4)$$

where

$$\mathcal{L}_E^f = -\bar{N} \left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + (m_N^0 - \mu) \right] N + C_S f \bar{N} N + \frac{1}{2} C_S f^2. \quad (5)$$

Since $C_S < 0$, the f integration is convergent.

Let M be the nucleon matrix. M has the block diagonal form,

$$M = M_{\text{block}} \oplus M_{\text{block}} \oplus M_{\text{block}} \oplus M_{\text{block}}, \quad (6)$$

where we have one block for each of the four nucleon states and

$$M_{\text{block}} = -\left[\partial_4 - \frac{\vec{\nabla}^2}{2m_N} + (m_N^0 - \mu) \right] + C_S f. \quad (7)$$

We note that M is real valued and therefore $\det M \geq 0$.

Consider the two-nucleon operator $A_2(x) = [N]_i [N]_j(x)$, where $i \neq j$ are indices for two different nucleon states. The two-point correlation function for A_2 is

$$\langle A_2(x) A_2^\dagger(0) \rangle_{\mu, T} = \langle [N]_i [N]_j(x) [N^*]_j [N^*]_i(0) \rangle_{\mu, T}. \quad (8)$$

Using our Euclidean functional integral representation, we have

$$\langle A_2(x) A_2^\dagger(0) \rangle_{\mu, T} = \int D\Theta [M_{\text{block}}^{-1}(x, 0)]^2 \quad (9)$$

where $D\Theta$ is the positive normalized measure defined by

$$D\Theta = \frac{Df \det M \exp(\frac{1}{2} C_S \int d^4x f^2)}{\int Df \det M \exp(\frac{1}{2} C_S \int d^4x f^2)}. \quad (10)$$

We note that since M_{block} is real valued, M_{block}^{-1} is also real valued.

Next we consider the three-nucleon and four-nucleon operators $A_3(x) = [N]_i [N]_j [N]_k(x)$ and $A_4(x) = [N]_i [N]_j [N]_k [N]_l(x)$, where i, j, k, l are all distinct. We have

$$\langle A_{3,4}(x) A_{3,4}^\dagger(0) \rangle_{\mu, T} = \int D\Theta [M_{\text{block}}^{-1}(x, 0)]^{3,4}. \quad (11)$$

We note that

$$\begin{aligned} \int D\Theta |M_{\text{block}}^{-1}(x, 0)|^3 &= \int D\Theta |M_{\text{block}}^{-1}(x, 0)| [M_{\text{block}}^{-1}(x, 0)]^2 \\ &\leq \sqrt{\int D\Theta [M_{\text{block}}^{-1}(x, 0)]^2} \\ &\quad \times \sqrt{\int D\Theta [M_{\text{block}}^{-1}(x, 0)]^4}, \end{aligned} \quad (12)$$

where the second line is from the Cauchy-Schwarz inequality. Therefore

$$|\langle A_3(x) A_3^\dagger(0) \rangle_{\mu, T}| \leq \sqrt{\langle A_2(x) A_2^\dagger(0) \rangle_{\mu, T} \langle A_4(x) A_4^\dagger(0) \rangle_{\mu, T}}. \quad (13)$$

Let E_{A_2} be the binding energy of the lowest state that couples to A_2 , and E_{A_4} be the binding energy of the lowest state that couples to A_4 . From stability arguments we know that

$$E_{A_2} \geq E_{A_3} \geq E_{A_4}, \quad 2E_{A_2} \geq E_{A_4}, \quad (14)$$

since otherwise the three-nucleon and four-nucleon states could decay into nucleons or smaller nuclei. But now we can use (13) to prove a different inequality. Taking the limit $x \rightarrow \infty$ in the temporal direction we see that any state with the quantum numbers of A_3 must have energy less than the average of E_{A_2} and E_{A_4} ,

$$E_{A_3} \geq \frac{1}{2} [E_{A_2} + E_{A_4}]. \quad (15)$$

Taking the limit $x \rightarrow \infty$ in any spatial direction we have that the inverse correlation length for A_3 must be greater than the average of the inverse correlation lengths for A_2 and A_4 ,

$$\xi_{A_3}^{-1} \geq \frac{1}{2}[\xi_{A_2}^{-1} + \xi_{A_4}^{-1}]. \quad (16)$$

Using arguments similar to those in [5], we can show that the inequalities (15) and (16) hold for a general three-nucleon operator

$$A_3(x) = \int_{\Omega} d^4x_1 d^4x_2 a_{ijk}(x_1, x_2) [N]_i(x + x_1) \times [N]_j(x + x_2) [N]_k(x + x_3), \quad (17)$$

so long as Ω is bounded.

In the real world C_{odd} is small but nonzero. We can measure the shift in the energy of a given state $|A\rangle$ using first-order perturbation theory, $\Delta E_A = \langle A|H'|A\rangle$, where

$$H' = \frac{1}{2} C_{\text{odd}} \int d^3\vec{x} [\bar{N} \vec{\sigma} N \cdot \bar{N} \vec{\sigma} N - \bar{N} \vec{\tau} N \cdot \bar{N} \vec{\tau} N]. \quad (18)$$

Let us first consider two-nucleon states in an S wave. In terms of $SU(4)$ representations, we decompose the tensor product of two fundamental four-dimensional representations, $4 \otimes 4 = 10 \oplus 6$. The six-dimensional representation is antisymmetric, and the spin and isospin representations must be $6 = (1, 0) \oplus (0, 1)$. The spin triplet with isospin singlet corresponds with the deuteron, D , while the spin singlet with isospin triplet corresponds with the nearly bound 1S_0 states. A similar analysis for S -wave three-nucleon states gives us one antisymmetric $\bar{4}$ representation with spin-isospin content $\bar{4} = (\frac{1}{2}, \frac{1}{2})$. This multiplet corresponds with the triton, T , and Helium-3. There is only one antisymmetric S -wave four-nucleon state. It is a therefore spin singlet and isospin singlet and corresponds with Helium-4.

Under a transformation that interchanges spin and isospin degrees of freedom, $|{}^4\text{He}\rangle$ is mapped into itself, possibly with a minus sign. However, H' is odd under this transformation and therefore, $\langle {}^4\text{He}|H'|{}^4\text{He}\rangle = 0$. Under the interchange of spin and isospin, the spin-up ${}^3\text{He}$ state and spin-down T state are also mapped into themselves, possibly with minus signs. We conclude that $\langle {}^3\text{He}|H'|{}^3\text{He}\rangle = \langle T|H'|T\rangle = 0$. Under the interchange of spin and isospin, the D states and 1S_0 states interchange with each other, again possibly with minus signs. Therefore, $\langle D|H'|D\rangle = -\langle {}^1S_0|H'|{}^1S_0\rangle$. We can now adjust for the first-order energy corrections due to H' ,

$$E_{3\text{He}}, E_T \geq \frac{1}{2} \left[\frac{1}{2} (E_D + E_{1S_0}) + E_{4\text{He}} \right]. \quad (19)$$

The physical binding energies are shown in Table II [17]. Plugging these values into (19), we find that the inequality is satisfied, -7.7 MeV , $-8.5 \text{ MeV} \geq -14.7 \text{ MeV}$. An analogous relation can be derived for the inverse correlation lengths,

$$\xi_{3\text{He}}^{-1}, \xi_T^{-1} \geq \frac{1}{2} \left[\frac{1}{2} (\xi_D^{-1} + \xi_{1S_0}^{-1}) + \xi_{4\text{He}}^{-1} \right]. \quad (20)$$

TABLE II. Binding energies for light nuclides.

| | $\sim 0 \text{ MeV}$ (nearly bound) |
|-----------------|-------------------------------------|
| 1S_0 | |
| D | -2.224 MeV |
| ${}^3\text{He}$ | -7.718 MeV |
| T | -8.481 MeV |
| ${}^4\text{He}$ | 28.296 MeV |

We stress that all of these inequalities hold in symmetric nuclear matter at any density or temperature where the effective theory description is valid. From here on we will consider only energy inequalities since the inverse correlation length inequalities are completely analogous.

We now generalize the results (15) and (16) to the case where the Wigner symmetry is a $SU(N)$ symmetry, for arbitrary N . Let n_{small} , n_{big} , and n be any integers such that $0 \leq 2n_{\text{small}} < n < 2n_{\text{big}} \leq N$. Then

$$E_n \geq \frac{n - 2n_{\text{small}}}{2n_{\text{big}} - 2n_{\text{small}}} E_{2n_{\text{big}}} + \frac{2n_{\text{big}} - n}{2n_{\text{big}} - 2n_{\text{small}}} E_{2n_{\text{small}}}. \quad (21)$$

This inequality is a statement of convexity of the energy as a function of nucleon number, with the additional requirements that the number of nucleons is less than or equal to N and the two end points have an even number of nucleons. The proof of the inequality is straightforward. We can write

$$\int D\Theta |M_{\text{block}}^{-1}(x, 0)|^n = \int D\Theta |M_{\text{block}}^{-1}(x, 0)|^z |M_{\text{block}}^{-1}(x, 0)|^{n-z}. \quad (22)$$

Applying the Hölder inequality to the right-hand side, one finds the upper bound (21).

Taking $N = 4$ and setting $2n_{\text{small}} = 2$, $n = 3$, $2n_{\text{big}} = 4$, we recover (15). If, however, we let $2n_{\text{small}} = 0$, $n = 3$, $2n_{\text{big}} = 4$, we get $E_3 \geq \frac{3}{4} E_4$. There are no first-order H' corrections in this case, and we see that in the real world, this inequality is satisfied -7.7 MeV , $-8.5 \text{ MeV} \geq -21.2 \text{ MeV}$. Setting $2n_{\text{small}} = 0$, $n = 2$, $2n_{\text{big}} = 4$, we get $E_2 \geq \frac{1}{2} E_4$. With first-order H' corrections we have

$$\frac{1}{2} (E_D + E_{1S_0}) \geq \frac{1}{2} E_{4\text{He}}, \quad (23)$$

and this is also satisfied, $-1.1 \text{ MeV} \geq -14.1 \text{ MeV}$. We note, however, that this was already known from the stability condition (14).

Up to this point we have ignored three- and four-nucleon forces. It has been shown that the dominant three-nucleon force is Wigner-symmetric [8,13]. We now show that introducing Wigner-symmetric three- and four-nucleon forces does not spoil positivity of the Euclidean functional integral so long as the three-nucleon force is not too strong and the four-nucleon force is not too repulsive. We want to find a Hubbard-Stratonovich trans-

formation that reproduces a contribution to the action of the form,

$$\prod_x \exp[c_2(\bar{N}N(x))^2 + c_3(\bar{N}N(x))^3 + c_4(\bar{N}N(x))^4]. \quad (24)$$

Let us just concentrate on what happens at a single point x , and in our notation, we suppress writing the x explicitly. We note that $\bar{N}N$ raised to any power greater than four must vanish. So we have

$$\begin{aligned} & \exp[c_2(\bar{N}N)^2 + c_3(\bar{N}N)^3 + c_4(\bar{N}N)^4] \\ &= a_0 + a_1\bar{N}N + \frac{a_2}{2!}(\bar{N}N)^2 + \frac{a_3}{3!}(\bar{N}N)^3 + \frac{a_4}{4!}(\bar{N}N)^4, \end{aligned} \quad (25)$$

$$\begin{aligned} a_0 &= 1, & a_1 &= 0, & a_2 &= 2c_2, \\ a_3 &= 6c_3, & a_4 &= 12c_2^2 + 24c_4. \end{aligned} \quad (26)$$

We now to try find a real function $g(f)$ such that

$$\begin{aligned} \int_{-\infty}^{\infty} df \exp[f\bar{N}N + g(f)] &= a_0 + a_1\bar{N}N + \frac{a_2}{2!}(\bar{N}N)^2 \\ &+ \frac{a_3}{3!}(\bar{N}N)^3 + \frac{a_4}{4!}(\bar{N}N)^4. \end{aligned} \quad (27)$$

We observe that a Hubbard-Stratonovich transformation of this form maintains the positive functional integral measure. Expanding the left-hand side, we have

$$a_n = \int_{-\infty}^{\infty} df f^n \exp[g(f)], \quad n = 0, 1, 2, 3, 4. \quad (28)$$

Finding sufficient and necessary conditions for the existence of $g(f)$ is known in the mathematics literature as the truncated Hamburger moment problem. This problem has been solved [18], and in our case $g(f)$ exists if and only if the so-called block-Hankel matrix,

$$\begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2c_2 \\ 0 & 2c_2 & 6c_3 \\ 2c_2 & 6c_3 & 12c_2^2 + 24c_4 \end{bmatrix}, \quad (29)$$

is positive semidefinite, with the added condition that if $c_2 = 0$ then $c_4 = 0$. The determinant of this matrix is $16c_2^3 - 36c_2^2c_3 + 48c_2c_4$. With an attractive two-nucleon force and small three- and four-nucleon forces, the conditions are clearly satisfied. Whether or not these conditions are satisfied in the real world and at which lattice spacings is beyond the scope of this Letter. But hopefully this will be numerically determined in the near future.

To summarize, the physics of low-energy symmetric nuclear matter is close to the Wigner limit. We have proven that the energy of any three-nucleon state is bounded below by the average energy of the lowest two-nucleon and four-nucleon states. We have calculated the corrections due to the lowest-order terms breaking Wigner symmetry and shown that the inequalities are satisfied.

We have proven general energy convexity results for $SU(N)$ and shown that all of these inequalities are satisfied for $N = 4$. We have also discussed the inclusion of Wigner-symmetric three- and four-nucleon forces.

The authors thank John Thomas for discussions and Paulo Bedaque for his hospitality at Lawrence Berkeley Laboratory where part of this research was completed. This work was supported by the U.S. Department of Energy under Grants No. DE-FG-88ER40388 and No. DE-FG02-04ER41335 and the National Science Council of Taiwan, ROC.

-
- [1] S. Weinberg, Phys. Lett. B **251**, 288 (1990).
 - [2] C. Ordonez, L. Ray, and U. van Kolck, Phys. Rev. C **53**, 2086 (1996); D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. **B478**, 629 (1996); **B534**, 329 (1998); S. R. Beane, P. F. Bedaque, W. C. Haxton, D. R. Phillips, and M. J. Savage, nucl-th/0008064; P. F. Bedaque and U. van Kolck, Annu. Rev. Nucl. Part. Sci. **52**, 339 (2002); E. Epelbaum, W. Gloeckle, and U.-G. Meissner, Eur. Phys. J. A **19**, 401 (2004); nucl-th/0405048.
 - [3] H. M. Müller, S. E. Koonin, R. Seki, and U. van Kolck, Phys. Rev. C **61**, 044320 (2000); J.-W. Chen and D. B. Kaplan, Phys. Rev. Lett. **92**, 257002 (2004); D. Lee, B. Borasoy, and T. Schafer, Phys. Rev. C **70**, 014007 (2004); T. Abe, R. Seki, and A. N. Kocharian, nucl-th/0312125.
 - [4] N. Kaiser, S. Fritsch, and W. Weise, Nucl. Phys. **A724**, 47 (2003); A. Bhattacharyya and R. J. Furnstahl, nucl-th/0408014; A. Schwenk and J. Polonyi, nucl-th/0403011.
 - [5] D. Lee, nucl-th/0407088.
 - [6] D. Lee, nucl-th/0407101.
 - [7] D. Weingarten, Phys. Rev. Lett. **51**, 1830 (1983); C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984); S. Nussinov and M. A. Lampert, Phys. Rep. **362**, 193 (2002); T. D. Cohen, Phys. Rev. Lett. **91**, 032002 (2003).
 - [8] T. Mehen, I. W. Stewart, and M. B. Wise, Phys. Rev. Lett. **83**, 931 (1999).
 - [9] E. Epelbaum, U. G. Meissner, W. Gloeckle, and C. Elster, Phys. Rev. C **65**, 044001 (2002).
 - [10] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. **92**, 150402 (2004); C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. **92**, 040403 (2004).
 - [11] M. Lüscher, Commun. Math. Phys. **105**, 153 (1986).
 - [12] S. R. Beane, P. F. Bedaque, A. Parreno, and M. J. Savage, hep-lat/0312004.
 - [13] P. F. Bedaque, H. W. Hammer, and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999); Nucl. Phys. **A676**, 357 (2000).
 - [14] E. Wigner, Phys. Rev. **51**, 106 (1939).
 - [15] D. B. Kaplan and M. J. Savage, Phys. Lett. B **365**, 244 (1996).
 - [16] D. B. Kaplan and A. V. Manohar, Phys. Rev. C **56**, 76 (1997).
 - [17] G. Audi and A. H. Wapstra, Nucl. Phys. **A595**, 409 (1995).
 - [18] V. Adamyan, J. Alcober, and I. Tkachenko, American Mathematics Research eXpress **2**, 33 (2003); R. Curto and L. Fialkow, Houston J. Math. **17**, 603 (1991).