Neutrino Oscillations in Low Density Medium

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We have solved the evolution equation for neutrinos in a low density medium, $V \ll \Delta m^2/2E$, where V is the matter potential, developing the perturbation theory in $\epsilon \equiv 2VE/\Delta m^2$. Simple and physically transparent formulas for the oscillation probabilities have been derived in the lowest order in ϵ which are valid for an arbitrary density profile. The formulas can be applied for propagation of the solar and supernova neutrinos in matter of the Earth, substantially simplifying numerical calculations. Using these formulas we study sensitivity of the oscillations to structures of the density profile situated at different distances d from a detector. For the mass-to-flavor transitions, e.g., $\nu_2 \to \nu_e$, we have found the attenuation effect: a decrease of the sensitivity to remote structures, $d > l_\nu E/\Delta E$, where l_ν is the oscillation length and $\Delta E/E$ is the energy resolution of a detector.

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Introduction.—Oscillations of the low energy neutrinos in matter of the Earth [1–3] have a number of implications of the fundamental importance.

(i) For solar neutrinos detection of the day-night asymmetry and measurements of the zenith and energy dependences of the regeneration effect [1–13] are crucial for consistency checks of the large mixing angle (LMA) Mikheyev-Smirnov-Wolfenstein solution to the solar neutrino problem. The asymmetry and regeneration are the key (still unestablished) signatures of this solution. Furthermore, their detection will allow one to better determine the oscillation parameters, and to search for effects of the 1–3 mixing [14], as well as nonstandard neutrino interactions.

(ii) Oscillations in the Earth lead to distortion of the neutrino spectra from supernovae and to difference of signals in different detectors [15–22]. Using two or more detectors one can perform the "long-baseline experiments" with supernova neutrinos avoiding the astrophysical uncertainties. This opens a possibility to establish the type of neutrino mass hierarchy [15,16], to measure the oscillation parameters, and to determine characteristics of the original neutrino fluxes.

(iii) In the future, the solar and supernova neutrinos can be used for tomography of the Earth (see, e.g., [20,21]) and for searches for small structures of the density profile [20].

Previously, the Earth matter effects were studied [1–20] in the approximation of one or several layers (mainly, mantle and core) with constant densities or by performing direct numerical integration of the evolution equation. However, simple analytic methods elaborated mainly for the high energy neutrinos (atmospheric and accelerator) do not work at low energies when the oscillation length becomes comparable with the size of small structures of the density profile. Numerical calculations are extremely lengthy, and, moreover, results of different groups do not agree with each other.

In this Letter we derive a very precise formula for the oscillation probability, which simplifies substantially numerical calculations and allows one to understand all features of the oscillation effects in details. The formula is the generalization of our result in [20] obtained for thin layers of matter.

ϵ-perturbation theory.—For the LMA oscillation parameters oscillations of the solar and supernova neutrinos inside the Earth occur in the weak matter effect regime, when the matter potential, V, is much smaller than the kinetic energy of the neutrino system:

$$V \ll \Delta m^2/2E. \tag{1}$$

Here $V(x) \equiv \sqrt{2} G_F N_e(x)$, G_F is the Fermi constant, $N_e(x)$ is the number density of the electrons, and $\Delta m^2 \equiv m_2^2 - m_1^2$ is the mass squared difference. In this case one can introduce a small parameter

$$\epsilon(x) \equiv \frac{2EV(x)}{\Delta m^2} \approx 0.02 \left[\frac{E}{10\,\mathrm{MeV}}\right]\left[\frac{N_e(x)}{N_A}\right]\left[\frac{7\times 10^{-5}\,\mathrm{eV}^2}{\Delta m^2}\right], \tag{2}$$

where N_A is the Avogadro number, and develop the perturbation theory in $\epsilon(x)$. (For $E = 10$ MeV we find $\epsilon \sim$ 0.03–0.1 inside the Earth.)

Let us consider two active neutrino mixing $\nu_f = U(\theta)\nu_{\mathrm{mass}}$, where $\nu_f \equiv (\nu_e, \nu_a)^T$ and $\nu_{\mathrm{mass}} \equiv (\nu_1, \nu_2)^T$ are the flavor and mass states correspondingly; ν_a is a combination of ν_μ and ν_τ, θ is the mixing angle in vacuum, and the mixing matrix is defined as

$$U(\theta) \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{3}$$

First, we find the evolution matrix for the mass states and then make projection onto the flavor states. The evolution equation is given by

$$i\frac{d\nu_{\mathrm{mass}}}{dx} = H(x)\nu_{\mathrm{mass}} \tag{4}$$

with the Hamiltonian

$$H(x) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\Delta m^2}{2E} \end{pmatrix} + U^\dagger \begin{pmatrix} V(x) & 0 \\ 0 & 0 \end{pmatrix} U. \tag{5}$$

The Hamiltonian can be rewritten as [23]

$$H(x) = U'(x)\begin{pmatrix} 0 & 0 \\ 0 & \Delta_m(x) \end{pmatrix} U'^\dagger(x), \tag{6}$$

where

$$\Delta_m(x) \equiv \frac{\Delta m^2}{2E}\sqrt{[\cos 2\theta - \epsilon(x)]^2 + \sin^2 2\theta} \tag{7}$$

is the difference of energies of the neutrino eigenstates in matter $\nu_m \equiv (\nu_{1m}, \nu_{2m})^T$, and $U' \equiv U(\theta')$ is the instantaneous mixing matrix of the mass states in matter with the angle θ' determined by

$$\sin 2\theta' = \frac{\epsilon \sin 2\theta}{\sqrt{(\cos 2\theta - \epsilon)^2 + \sin^2 2\theta}} = \epsilon \sin 2\theta_m. \tag{8}$$

Here θ_m is the mixing angle of the flavor states in matter. Taking into account relations $\nu_f = U(\theta_m)\nu_m$ and $\nu_{\mathrm{mass}} = U(\theta')\nu_m$, we find that $\nu_f = U_m U'^\dagger \nu_{\mathrm{mass}}$, and, consequently, $U = U_m U'^\dagger$ and $\theta' = \theta_m - \theta$. According to (8) the angle θ' is small: $\sin 2\theta' \leq \epsilon$.

The formal solution of Eq. (4), that is, the evolution matrix for the mass eigenstates from the initial point x_0 to the final point x_f, can be written as

$$S_{x_0 \to x_f} = T e^{-i\int_{x_0}^{x_f} H(x)dx}, \tag{9}$$

where T means the chronological ordering. Let us divide a trajectory of neutrinos into n equal parts (layers) of the size Δx, so that $n = (x_f - x_0)/\Delta x$, and assume constant density inside each layer. Then according to (6) for the evolution matrix we obtain

$$S = T\prod_{j=n\ldots 1} e^{-iH(x_j)\Delta x} = T\prod_{j=n\ldots 1} U'_j D_j U'^\dagger_j \tag{10}$$

($x_n \equiv x_f$), where

$$D_j \equiv \mathrm{diag}(1, e^{-i\phi_j^m}), \qquad \phi_j^m \equiv \Delta_m(V_j)\Delta x \tag{11}$$

is the evolution matrix of the matter eigenstates in the jth layer, ϕ_j^m is the relative phase between the matter eigenstates acquired in the layer j, and $V_j \equiv V(x_j)$ is the value of the potential in the jth layer. $U'_j \equiv U(\theta'_j)$ is the mixing matrix of the mass states in the layer j.

For the jth block we obtain the expression

$$U'_j D_j U'^\dagger_j = D_j + G_j, \tag{12}$$

where

$$G_j = (e^{-i\phi_j} - 1)\left[\frac{\sin 2\theta'_j}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin^2\theta'_j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]. \tag{13}$$

Inserting (12) into (10) we obtain an expression for the S matrix as a power series of $G_j = O(\epsilon)$.

In the limit $n \to \infty$ and $\Delta x \to 0$ the sums are substituted by the integrals:

$$\prod_{j=k\ldots n} \exp(-i\phi_j^m) \to \exp(-i\phi^m_{x_k \to x_n}), \tag{14}$$

where, in general,

$$\phi^m_{a \to b} \equiv \int_a^b dx \Delta_m(x). \tag{15}$$

Furthermore, $G_j \to -i\Upsilon(x)dx$, and, according to (13),

$$\Upsilon = \frac{\sin 2\theta}{2} V(x)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \Delta_m(x)\sin^2\theta'(x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{16}$$

Here we have taken into account the relation (8).

In terms of Υ the S matrix can be written as

$$\begin{aligned} S = {} & \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x_0 \to x_f}} \end{pmatrix} - i\int_{x_0}^{x_f} dx \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x \to x_f}} \end{pmatrix} \Upsilon(x) \\ & \times \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x_0 \to x}} \end{pmatrix} - \int_{x_0}^{x_f} dx \int_{x_0}^{x} dy \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x \to x_f}} \end{pmatrix} \\ & \times \Upsilon(x)\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{y \to x}} \end{pmatrix}\Upsilon(y)\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x_0 \to y}} \end{pmatrix} + \cdots. \end{aligned} \tag{17}$$

Till now we did not make any approximation. Taking the zero and the first order terms in Υ in (17) and dropping the $\sin^2\theta' = O(V^2)$ term in (16), we obtain

$$\begin{aligned} S = {} & \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\phi^m_{x_0 \to x_f}} \end{pmatrix} - i\frac{\sin 2\theta}{2}\int_{x_0}^{x_f} dx V(x) \\ & \times \begin{pmatrix} 0 & e^{-i\phi^m_{x_0 \to x}} \\ e^{-i\phi^m_{x \to x_f}} & 0 \end{pmatrix}. \end{aligned} \tag{18}$$

The key feature of our method is that we calculate the adiabatic phase exactly (without ϵ expansion), whereas the mixing and the amplitude of oscillations are found approximately using the perturbation in ϵ. This allows one to apply results for an arbitrary length of the neutrino trajectory L. By making the ϵ expansion of the phase one should require that the error $\Delta\phi(L) \ll 2\pi$. Then the zero order term in Δ_m can be used for distances $L \ll l_\nu/\epsilon$ (which is not satisfied for the Earth crossing trajectories); the zero and first order terms give precise results for $L \ll l_\nu/\epsilon^2$, etc.

Notice that the second term in Υ (16) (of the order ϵ^2) is proportional to the diagonal matrix. This term does not contain an oscillatory factor and therefore, being inserted in (17), produces the contribution to the S matrix, which is proportional to the length of trajectory. It can be shown that this large contribution is canceled exactly by the contribution from the term of the second order in Υ.

An interesting feature of the formula (18), which leads to important consequences, is that in the second term the 12-element is determined by the phase acquired from the initial point x_0 to a given point x, whereas the 21-element depends on the phase from a given point x to the final point of evolution x_f.

Probabilities of neutrino conversion.—Using the evolution matrix for the mass states (18) we can calculate the amplitudes of various transitions. The evolution matrix in the flavor basis, S_f, can be obtained as $S_f = USU^\dagger$, where U is the vacuum mixing matrix (3).

The solar and supernova neutrinos arrive at the surface of the Earth as the incoherent fluxes of the mass eigenstates. So, for their detection the mass-to-flavor transitions, $\nu_i \to \nu_\alpha$, should be considered. In this case the evolution matrix equals $S_{fm} = US$, and the amplitude of the mass-to-flavor transition is given by

$$A_{\nu_i \to \nu_\alpha} = U_{\alpha j}(\theta) S_{ji}. \tag{19}$$

Using S_{ji} from (18) we find the probability of the $\nu_2 \to \nu_e$ transition, $P_{\nu_2 \to \nu_e} = |A_{\nu_2 \to \nu_e}|^2 = |U_{ej}(\theta) S_{j2}|^2$:

$$P_{\nu_2 \to \nu_e} = \sin^2\theta + \frac{1}{2}\sin^2 2\theta \int_{x_0}^{x_f} dx V(x) \sin\phi^m_{x \to x_f}. \tag{20}$$

For the probability of the $\nu_1 \to \nu_e$ oscillations we have $P_{\nu_1 \to \nu_e} = 1 - P_{\nu_2 \to \nu_e}$. The regeneration parameter equals $f_{\rm reg} \equiv P_{\nu_2 \to \nu_e} - \sin^2\theta$ (see, e.g., [10]). Notice that oscillations of solar and supernova neutrinos inside the Earth are a pure matter effect.

The formula (20) is valid for an arbitrary density profile provided that the condition (1) is fulfilled. It is straightforward to show that they reproduce all known results for particular density distributions: for one layer with constant density, for n symmetric shells [24], etc. The formula (20) can be obtained from the adiabatic perturbation theory [24], performing integration by parts in Eq. (18) of [24]. In the lowest order in ϵ the adiabatic perturbation theory and the ϵ-perturbation theory coincide.

Attenuation effect.—Let us use the formula (20) to estimate the sensitivity of the oscillation effect to various structures of density profile. Consider a structure of $V(x)$ in the point x. According to (20), this structure is integrated with the periodic function $\sin\phi_{x \to x_f}$ whose phase is acquired from the structure position to the detector: $d \equiv x_f - x$. The larger the distance, and therefore the phase, the stronger the averaging. Consequently, the effect of remote structures of the profile on the mass-to-flavor oscillation probabilities is suppressed.

To quantify this attenuation effect let us introduce the energy resolution function $f(E', E)$ and average the probability folded with f:

$$\overline{P}_{\nu_2 \to \nu_e} = \int dE' f(E', E) P_{\nu_2 \to \nu_e}. \tag{21}$$

The averaging effect can be parametrized by the *attenuation* factor $F(d)$ in the probability defined as

$$\overline{P}_{\nu_2 \to \nu_e} = \sin^2\theta + \frac{\sin^2 2\theta}{2} \int_{x_0}^{x_f} dx V(x) F(x_f - x) \times \sin\phi^m_{x \to x_f}, \tag{22}$$

so that in the absence of averaging $F = 1$.

For simplicity let us take the boxlike resolution function $f(E', E)$ for which

$$\overline{P}_{\nu_2 \to \nu_e} = \frac{1}{\Delta E} \int_{E-(\Delta E/2)}^{E+(\Delta E/2)} dE P_{\nu_2 \to \nu_e}. \tag{23}$$

Assuming that $\Delta E \ll E$ and approximating $\Delta_m \simeq \Delta m^2[1 - \epsilon\cos 2\theta]/2E$, we find

$$F(d) = \frac{1}{Q(d)} \sin Q(d), \qquad Q(d) \equiv \frac{\pi d \Delta E}{l_\nu E}, \tag{24}$$

where $l_\nu \approx l_m$ is the oscillation length (see Fig. 1).

The factor $F(d)$ (Fig. 1) is a decreasing function of the distance from a detector, so that contributions from large distances to the integral (22) are suppressed. The larger ΔE, the smaller the width of the first peak of $F(d)$. As follows from (24), the width and therefore the region of nonsuppressed contributions are given by

$$d < l_\nu E/\Delta E. \tag{25}$$

The attenuation effect allows us to explain certain features of the zenith angle distribution of the regenera-

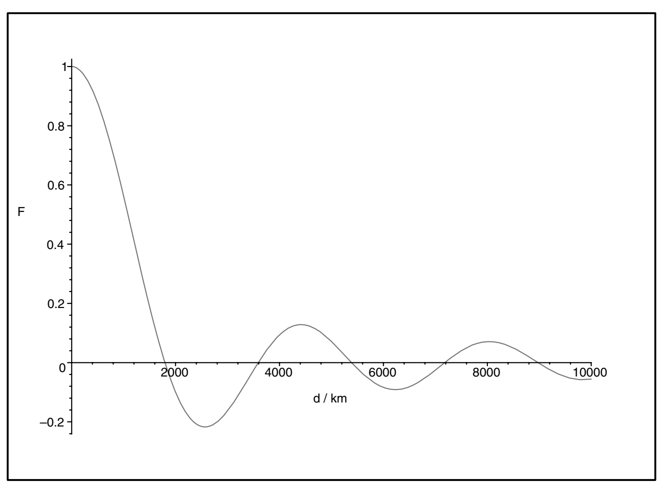

FIG. 1. The attenuation factor F as a function of $d \equiv x_f - x$ for $E = 10$ MeV, $\Delta E = 2$ MeV, and $\Delta m^2 = 7 \times 10^{-5}$ eV^2.

tion factor. Numerical computations do not show significant enhancement of the regeneration for the core crossing trajectories in comparison with that for the trajectories obtained in the mantle only, in spite of the fact that the density of core, and therefore ϵ, is 2–3 times larger (see, e.g., [10,11,13]). The explanation is straightforward: the border of the core is about $D = 3000$ km from the surface of the Earth. According to Fig. 1, the effect of structures from such a distance is suppressed by a factor of 3–5, which compensates for the increase of ϵ. At the same time, the effect of small structures near the surface of the Earth can be substantial. Different parametrizations of these structures used in numerical calculations lead to different results.

For comparison let us consider the inverse flavor-to-mass transition, e.g., $\nu_e \to \nu_2$. The probability $P_{\nu_e \to \nu_2}$ has a similar expression as in (20) but with the phase acquired on the way from the initial point x_0 to the position of structure x: $\phi_{x_0 \to x}$. Therefore with the flavor-to-mass transition one would probe structures at the opposite (to the detector) side of the Earth. This general consideration is in agreement with our results for thin layers [20].

For the flavor-to-flavor transition we obtain

$$P_{\nu_e \to \nu_\mu} = \sin^2 2\theta \bigg[\sin^2 \frac{1}{2} \phi^m_{x_0 \to x_f} + \frac{\cos 2\theta}{2} \int_{x_0}^{x_f} dx V(x) \times (\sin \phi^m_{x_0 \to x} + \sin \phi^m_{x \to x_f}) \bigg]. \tag{26}$$

The probability (26) is sensitive to structures near the surface from both sides of the Earth.

Conclusion.—We have found an approximate (but very precise) solution of the evolution equation for the low energy neutrinos in matter. The solution is valid for *arbitrary* density profile and length of neutrino trajectory. It has been obtained using improved perturbation theory in which the mixing (amplitude of oscillations) is found as a series expansion in ϵ, whereas the adiabatic oscillation phase is calculated exactly. We have found simple and physically transparent formula for the oscillation probabilities in the first order in ϵ. The results can be applied to the solar and supernova neutrinos crossing the Earth. They simplify substantially numerical calculations of the oscillation effects.

We have found the attenuation effect for the mass-to-flavor transitions according to which probabilities are sensitive to structures situated close enough to a detector, and the effect of the remote structures is suppressed. The distance that can be "viewed" by a detector is determined by the oscillation length and the energy resolution of a detector. These results can be used to explain the zenith angle dependence of the regeneration effect, and in the future for the oscillation tomography of the Earth.

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