Index Theorem and Universality Properties of the Low-Lying Eigenvalues of Improved Staggered Quarks

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(Received 9 June 2004; revised manuscript received 22 October 2004; published 6 December 2004)

We study various improved staggered quark Dirac operators on quenched gluon backgrounds in lattice QCD generated using a Symanzik-improved gluon action. We find a clear separation of the spectrum into would-be zero modes and others. The number of would-be zero modes depends on the topological charge as expected from the index theorem, and their chirality expectation value is large (≈ 0.7). The remaining modes have low chirality and show clear signs of clustering into quartets and approaching the random matrix theory predictions for all topological charge sectors. We conclude that improvement of the fermionic and gauge actions moves the staggered quarks closer to the continuum limit where they respond correctly to QCD topology.

DOI: 10.1103/PhysRevLett.93.241601 PACS numbers: 11.15.Ha, 12.38.Gc

Introduction.—It has been widely held that lattice staggered quarks are insensitive to the topology of the underlying gauge fields. The low-lying spectrum of the Dirac operator has neither shown the number of chiral (near-) zero modes anticipated from the index theorem, nor have the eigenvalues lain on the expected universal distributions.

Here we present evidence that this is not a generic failing of staggered quarks but simply a problem of discretization errors, and that the use of improved staggered Dirac operators clarifies the situation and points to the correct continuum behavior. This allows the use of improved staggered quarks in lattice QCD to study topologically sensitive states, such as those associated with the axial anomaly (principally the η' meson). It also has a bearing on establishing the effect of taking the fourth root of the staggered determinant to represent one flavor of staggered sea quarks.

We begin by reviewing our understanding in the continuum. The eigenmodes of the (anti-Hermitian, gauge covariant) massless Dirac operator are given by

$$\not{D} f_s = i\lambda_s f_s, \qquad \lambda_s \in \mathbb{R}, \tag{1}$$

where we use orthonormalized eigenvectors, $f_s^\dagger f_t = \delta_{st}$. As $\{\not{D}, \gamma_5\} = 0$, the spectrum is symmetric about zero: if $\lambda_s \neq 0$, then $\gamma_5 f_s$ is also an eigenvector with eigenvalue $-i\lambda_s$ and chirality $\chi_s \equiv f_s^\dagger \gamma_5 f_s = 0$. The zero modes, $\lambda_s = 0$, can be chosen with definite chirality: $\chi_s = \pm 1$. In general, there are $n_\pm$ such modes, whose relative number is fixed by the (gluonic) topological charge

$$Q = \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\sigma\tau} \mathrm{Tr} F_{\mu\nu}(x) F_{\sigma\tau}(x) \tag{2}$$

via the Atiyah-Singer index theorem [1,2]:

$$Q = m \mathrm{Tr} \frac{\gamma_5}{\not{D} + m} = n_+ - n_-, \tag{3}$$

where m is the quark mass. Based on [3], it has been suggested that (for a sufficiently large volume) the nonzero low-lying eigenmodes take values from a universal distribution [4] scaled by a QCD-specific quantity (the chiral condensate). The universality class is determined by the chiral symmetries of QCD with separate predictions for each sector of fixed topological charge. The distributions can be derived from any theory in the correct universality class, such as ensembles of random matrices [5,6] (for a review of other theories, see [7]).

The four-dimensional lattice staggered massless Dirac operator is anti-Hermitian, with a purely imaginary spectrum $i\lambda_s$. It represents $N_t = 4$ "tastes" of fermions that interact via highly virtual gluon exchange at finite lattice spacing, causing taste-symmetry violations [8]. These vanish in the continuum limit (as a^2), and we then expect to recover a fourfold degeneracy in the spectrum. A remnant of continuum chiral symmetry in the staggered action gives a local and taste-nonsinglet γ_5 operator that guarantees that the spectrum is symmetric about zero, as in the continuum. The γ_5 operator, γ_5^{ts}, relevant to the index theorem must be a taste-singlet one, however, since only this can couple to the vacuum correctly [9]. As $\{\not{D}, \gamma_5^{ts}\} \neq 0$, there is no exact index theorem, and all eigenmodes, in principle, contribute to Eq. (3) [10].

If the gauge field is sufficiently close to the continuum limit, however, we expect to see the continuum features developing. There should be $2|Q|$ near-zero modes on either side of zero, whose chiralities are close to unity. The taste-singlet γ_5^{ts} operator is not conserved, so we expect a renormalization to achieve a value of one in the continuum. The remaining modes should have chirality near zero, and come in approximately degenerate quartets on either side of zero. The values of the eigenvalue quartets should be described by the same universal distribution as continuum QCD, up to a renormalization of the chiral condensate.

This picture was not seen for thermalized lattices at finite lattice spacing using the simplest "one-link" (naïve) staggered operator with the unimproved Wilson gauge action. There was no clear separation (in eigenvalue or chirality) of near-zero modes of topological origin from the remainder of the spectrum [10–14]. The small eigenvalues did not follow the universal predictions, but in each sector of $Q \neq 0$ followed the distribution for $Q = 0$ [15–19]. (It should be noted that the method we follow here, of grouping the eigenvalues into quartets, was not followed because this feature of the spectrum was not evident.)

This failure can be ascribed to taste-changing interactions and the lack of a good continuum chiral symmetry in the one-link staggered Dirac operator: good agreement with predictions for all topological charge sectors has been seen for Dirac operators obeying the Ginsparg-Wilson relation [20–25].

Over the past few years considerable advances have been made in lattice QCD phenomenology through the use of so-called "improved staggered" fermion formulations [26]. The goal of this program is to systematically reduce the lattice artifact taste interactions, and we expect that this will make the chiral properties of the fermions more continuumlike. The variation of the topological susceptibility with the sea quark mass, in particular, shows that this is so: while it was largely insensitive to the presence of one-link staggered sea quarks [27–30], the improved operator gives a variation with m that agrees well with theoretical expectations [30,31]. It is thus pertinent to study in more detail the spectrum of the improved staggered Dirac operator, and we shall show here that this also now gives signs of converging to the expected results, in contrast to previous studies.

Results.—On a Euclidean lattice with lattice spacing a, the one-link massless staggered operator is

$$\not{D}(x, y) = \frac{1}{2a} \sum_{\mu=1}^{4} \eta_\mu(x)[U_\mu(x)\delta_{x+\hat{\mu},y} - \text{H.c.}], \quad (4)$$

with $\eta_1 = 1$ and $\eta_\nu = (-1)^{x_{\nu-1}} \eta_{\nu-1}$. We study also three improved operators: the ASQTAD [8,32–34], the HYP [35], and the FAT7XASQ [14,36]. These operators use "smeared" gauge fields in place of the U field above, obtained by multiplying U fields along combinations of bent paths from the start to end points of the original link. This reduces the coupling to highly virtual gluons and suppresses the taste-changing interactions. The ASQTAD action uses a "FAT7" smearing, which includes paths made of up to seven links, and tadpole improvement. The HYP action uses a hypercubic blocking procedure, involving reunitarization back onto SU(3), and the FAT7XASQ action uses two applications of the FAT7 smearing, and also includes reunitarization. The taste-changing interactions are most suppressed for the HYP and FAT7XASQ cases.

We calculate the eigenmodes for these Dirac operators on an ensemble of quenched (no sea quarks) SU(3) gauge configurations. The gauge action is Symanzik improved at tree level with tadpole improvement so that remaining discretisation errors from the gluon field are a small number times $\alpha_s a^2$. The majority of our results are from 1000 configurations on a periodic lattice with 16^4 sites and a lattice spacing of 0.093 fm [37], which represents a typical present day ensemble. The physical volume should be large enough that finite volume effects on the low-lying spectrum are negligible [25]. On each configuration, we determine the topological charge by two standard methods that involve cooling the gauge fields [31]. On 10% of the ensemble the two methods disagree on Q, but we have checked that this ambiguity makes no visible statistical difference to our results. We stress that cooling is used *solely* to determine Q; the eigenvalues are calculated on the "raw" configurations.

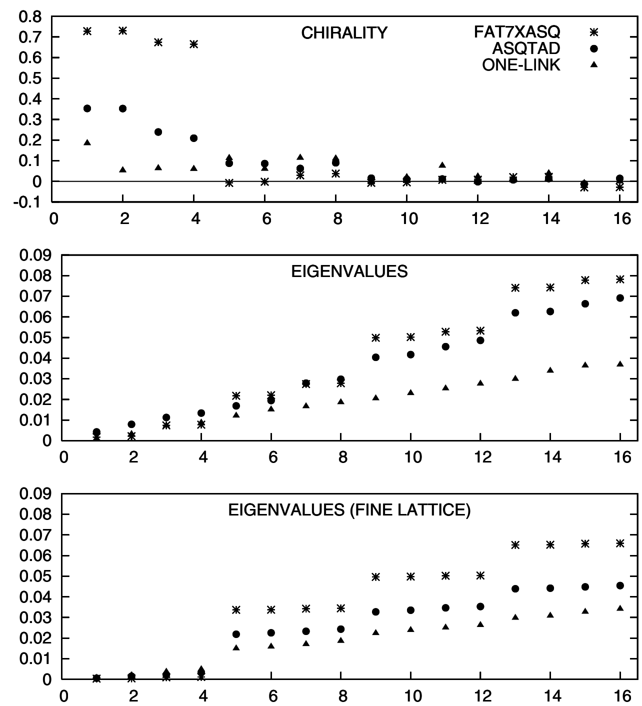

FIG. 1. The positive half of a typical low-lying eigenmode spectrum for a configuration of $Q = 2$, for various staggered fermion formulations from our $a = 0.093$ fm ensemble. The center panel gives the absolute value of the eigenvalue, λ_s, ordered according to increasing size. The x axis is then simply an eigenvalue number. The top panel is the chirality of the modes. The HYP action gives results very similar to FAT7XASQ and is not plotted for clarity. The bottom panel shows eigenvalue results for a $Q = 2$ configuration at $a = 0.077$ fm for comparison.

Figure 1 compares the low-lying modes of the spectra of the various staggered quark formulations on a typical background with $Q = 2$. We show the eigenvalue λ_s and chirality for the upper half of the spectrum. We see that the index theorem is well approximated for the more improved quark formulation. Specifically, there is a clear delineation between the near-zero modes (small eigenvalues and large chirality) and the rest of the spectrum. The number of near-zero modes is $2|Q|$, as expected. The other eigenvalues have small chirality and divide clearly into quartets. The taste-singlet γ_5^{ts} operator used for the chirality is a gauge-invariant point-split four-link operator.

The dependence on a and physical volume, V, of the small eigenvalues has been studied using additional ensembles. We find that, as expected, the would-be zero modes (converted to physical units) tend to zero as $a^2/\sqrt{V}$, and the nonzero modes as $1/V$. We will report elsewhere on this scaling analysis in more detail. We show in the bottom panel of Fig. 1, however, eigenvalues for a 20^4 configuration with $Q = 2$ from an ensemble with $a = 0.077$ fm. The features seen in the center panel become even more evident, with the separation of zero and nonzero modes now very clear.

As the near-zero modes for the more improved actions clearly separate and have well defined chirality, we may define an index $\bar{Q} = n_+ - n_-$ on each configuration. $\bar{Q}$ is then strongly correlated with Q. For the FAT7XASQ operator, if we count eigenmodes with absolute value of the chirality above 0.65 in $n_\pm$, for example, in about 90% of configurations for our $a = 0.093$ fm ensemble, $\bar{Q}$ and Q are the same, satisfying the index theorem. Indeed, measuring Q from the index theorem then becomes as reliable as measuring it from gluonic methods.

In Fig. 2, we show a scatter plot of the absolute value of the chirality versus the (absolute value of) eigenvalues, for different operators. We can see the formation of a gap between modes of small and large chirality as we improve the staggered operators, as well as the overall increase in chirality of the large chirality modes.

We turn now to the nonzero modes. Subtracting the $2|Q|$ near-zero modes from the spectrum, we group the other eigenvalues, ordered by size, into sets of four, as indicated in the FAT7XASQ and HYP cases in Fig. 1. We call the average of these sets $\Lambda_{1,2,\ldots}$. In Fig. 3, we plot the ratios $\langle\Lambda_s\rangle_Q/\langle\Lambda_t\rangle_Q$ (denoted by "s/t"), where the expectation values $\langle\cdot\rangle_Q$ are over the sectors with gluonic topological charges $\pm Q$ only. Also shown are the universal predictions. There is a clear dependence of the ratios on Q, in marked contrast with previous results, which showed no such variation. The results are systematically slightly lower than the theoretical predictions, especially for the ratios involving higher eigenvalues. This would be consistent with finite volume effects, as in [25].

There is also a small but systematic difference between the one-link and the improved actions, with the improved results showing a better agreement with the theoretical

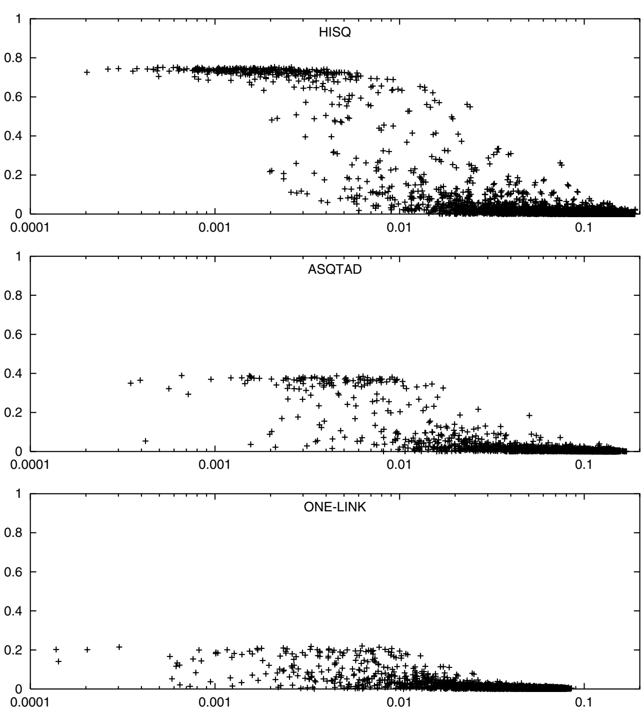

FIG. 2. A scatter plot for different staggered quark formulations, with absolute value of the chirality on the y axis and eigenvalue λ_s on the x axis. The lowest 50 eigenvalues for 147 configurations from our $a = 0.093$ fm ensemble are plotted.

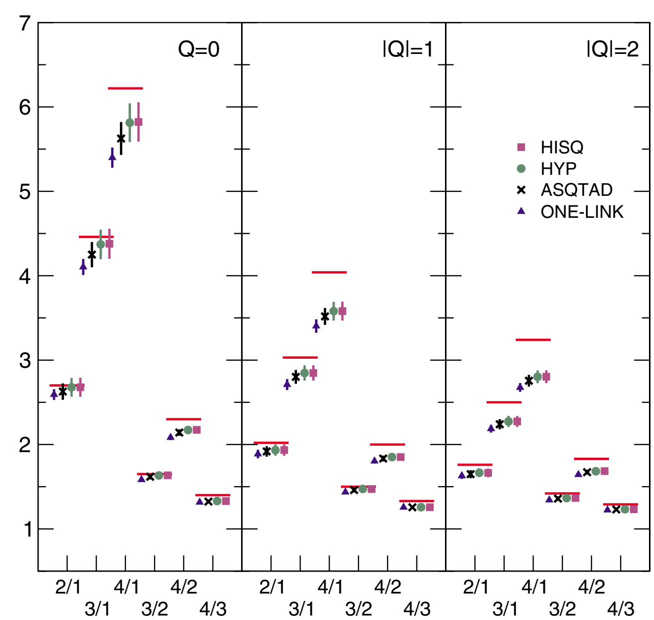

FIG. 3 (color online). The ratios of expectation values of small eigenvalues (see text for notation) for our $a = 0.093$ fm ensemble compared with the predictions based on a universal distribution (horizontal lines) for topological charge sectors 0, 1, and 2.

values. In agreement with [25] we find no significant changes on the coarse lattice at the same V.

Note that it is necessary to group the eigenvalues as explained above and cut out the near-zero modes to get sensible results. This is strong evidence that the four tastes are showing up in the spectrum, even where it is not directly evident in the spectrum itself. It seems clear from our analysis that results for unimproved staggered quarks on fine enough unimproved gluon fields would also show agreement with Q-dependent universal distributions and preliminary results confirm this.

Conclusions and outlook.—Improved staggered fermions are not blind to the topology, but, in fact, reproduce well the predictions of the index theorem and the universality of ratios of eigenvalues as a function of topological sector. This means we can have confidence in using them to attack the questions arising from the axial anomaly in QCD.

We also remark that the fact that the fourfold taste degeneracy of staggered quarks is becoming clear in the spectrum is encouraging for the program of establishing the effect of taking the fourth root of the staggered determinant to represent one flavor of staggered sea quarks. This program requires an analysis in the taste basis, and progress towards this is now possible.

More extensive studies of finite volume and lattice spacing effects and analysis of the eigenvectors are under way and will be reported elsewhere.

In the later stages of this study we became aware of work on a related topic [38].

We thank Ph. de Forcrand for his topological charge measurement code, A. Hasenfratz for help in implementing the HYP operator, and P. Lepage for many useful discussions. E. F. and C. T. H. D. are supported by PPARC and A. H. by the U.K. Royal Society. The eigenvalue calculations were carried out on computer clusters at Scotgrid and the Dallas Southern Methodist University. We thank David Martin and Kent Hornbostel for assistance.

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