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A New Geometric Invariant on Initial Data for the Einstein Equations

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For a given asymptotically flat initial data set for Einstein equations a new geometric invariant is constructed. This invariant measures the departure of the data set from the stationary regime; it vanishes if and only if the data are stationary. In vacuum, it can be interpreted as a measure of the total amount of radiation contained in the data.

Introduction.—The mass of an asymptotically flat data for Einstein equations measures the total amount of energy contained in the spacetime. The mass is zero if and only if the spacetime is flat. However, the energy can be in different forms: it can be in a stationary regime or in a dynamic one. This difference is, of course, physically important: gravitational radiation will be present only in the second case. The purpose of this Letter is to construct a new quantity that can measure how far the data are from the stationary regime. In other words, this quantity measures how dynamic the data are, and it will be zero if and only if the data are stationary. In vacuum, the dynamic is produced only by the gravitational field, then, in this case, the quantity can be interpreted as a measure of the total amount of radiation contained in the data.

The construction is based on a new definition of approximate symmetries. An approximate symmetry satisfies an equation that always has a solution for a generic spacetime and the solution coincides with the Killing vectors when the spacetime admits them. The new quantity is related to the approximate timelike translation.

There are many physical situations which are dynamic but nevertheless are expected to be described by data which are ''as close to stationary data as possible''; this invariant provides a precise meaning of this idea. An important example is the quasicircular orbit for binary black holes (see [1] and references therein). The invariant constructed here will provide a completely different way of finding this class of data and hopefully will help to solve the discrepancies between the different current approaches.

Symmetries, approximate symmetries, and the constraint map.—Let S be a 3-dimensional manifold, and let $\mathcal{M}_2, S_2, \mathcal{X}, C$ be the spaces of Riemannian metrics, symmetric 2-tensors, vectors, and scalar functions on S respectively. Let $h_{ab} \in \mathcal{M}_2$, and $K_{ab} \in S_2$. The constraint map $\Phi : \mathcal{M}_2 \times S_2 \to C \times \mathcal{X}$ is defined as follows

$$\Phi\begin{pmatrix} h_{ab} \\ K_{ab} \end{pmatrix} = \begin{pmatrix} R + K^2 - K_{ab}K^{ab} \\ -D^b K_{ab} + D_a K \end{pmatrix} \quad (1)$$

where D_a is the covariant derivative with respect to h_{ab}, $R = h^{ab}R_{ab}$, R_{ab} is the Ricci tensor of h_{ab}, $K = h^{ab}K_{ab}$, and $a, b, c...$ denote abstract indices; they are moved with

the metric h_{ab} and its inverse h_{ab}. The set (S, h_{ab}, K_{ab}) is called a vacuum initial data set for Einstein equations if $\Phi(h_{ab}, K_{ab}) = 0$ on S.

We can compute the linearization of Φ evaluated at (h_{ab}, K_{ab})

$$D\Phi\begin{pmatrix} \gamma_{ab} \\ Q_{ab} \end{pmatrix} = \begin{pmatrix} D^a D^b \gamma_{ab} - R_{ab}\gamma^{ab} - \Delta\gamma + H \\ -D^b Q_{ab} + D_a Q - F_a \end{pmatrix} \quad (2)$$

and its formal adjoint

$$D\Phi^*\begin{pmatrix} \eta \\ X^a \end{pmatrix} = \begin{pmatrix} D_a D_b \eta - \eta R_{ab} - \Delta\eta h_{ab} + H_{ab} \\ D_{(a}X_{b)} - D^c X_c h_{ab} + F_{ab} \end{pmatrix} \quad (3)$$

where $\gamma = \gamma_{ab}h^{ab}$, $Q = Q_{ab}h^{ab}$ and H; H_{ab}, F_a, and F_{ab} vanished when $K_{ab} = 0$ (since in this article we will only consider the time-symmetric case, the explicit expression of these quantities will not be needed).

The constraint map (1) is important not only because it characterizes the initial data for Einstein equations, but it also gives the Hamiltonian of the theory (see [2] and also [3,4]). In particular, the adjoint map (3) gives the right hand side of the evolution equations in the Hamiltonian formulation (see [5] and references therein). Moreover, $D\Phi^*$ has a remarkably property [6]: the elements of the kernel of $D\Phi^*$ are the *symmetries* of the spacetime determined by the initial data (S, h_{ab}, K_{ab}). That is, if (η, X^a) satisfies $D\Phi^*(\eta, X^a) = 0$ then the spacetime will have a Killing vector ξ, and (η, X^a) are the projections of ξ normal and tangential to the spacelike hypersurface S, respectively.

Motivated by this correspondence between the kernel of $D\Phi^*$ and symmetries, we introduce the concept of approximate symmetries. We will make use of the following related operator introduced by Bartnik [7]

$$\mathcal{P}\begin{pmatrix} \gamma_{cd} \\ q_{pcd} \end{pmatrix} = D\Phi\begin{pmatrix} \gamma_{cd} \\ -D^p q_{pcd} \end{pmatrix} \quad (4)$$

and its adjoint

$$\mathcal{P}\begin{pmatrix} \eta \\ X_a \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & D_p \end{pmatrix} \cdot D\Phi^*\begin{pmatrix} \eta \\ X_a \end{pmatrix}, \quad (5)$$

where the dot denotes matrix product. We say that (η, X^a) is an *approximate symmetry* if it satisfies the equation

$$\mathcal{P}\,\mathcal{P}^*(\eta, X^a) = 0 \qquad (6)$$

and has the falloff behavior at infinity of the Killing vectors in flat spacetime. Equation (6) is solved for a given (h_{ab}, K_{ab}). Note that although the composition $D\Phi D\Phi^*$ is formally well defined, there is a mismatch in the units in the domains and ranges of these operators; that is why we use $\mathcal{P}$ instead of $D\Phi$ in Eq. (6).

This definition will be meaningful if: (a) Every symmetry is also an approximate symmetry. (b) For generic data which admit no symmetry there exist always approximate symmetries. (c) We can uniquely associate each approximate symmetry with a symmetry in flat space, that is, we have approximate time translation, approximate boost, etc. Since every symmetry satisfies $D\Phi^*(\eta, X^a) = 0$, (a) trivially follows. In the next section we will prove that (b-c) are true for time-symmetric initial data. In order to simplify the analysis in this Letter we will assume that $K_{ab} = 0$ and $X^a \to 0$ at infinity; that is, we are only going to consider time translations and boosts (in the flat case we have four solutions). This is an important special case since it includes all the relevant features and difficulties of the equation. The general case will be studied in a subsequent article.

Equation (6) can be derived from a variational principle, it is the Euler-Lagrange equation of the following functional

$$J(\eta, X^a) = \int_S \mathcal{P}^*(\eta, X^a) \cdot \mathcal{P}^*(\eta, X^a) d\mu, \qquad (7)$$

where $d\mu$ is the volume element with respect to h_{ab}. For a symmetry we have $J = 0$, for an approximate symmetry $J \geq 0$. If J, evaluated at an approximate symmetry, is finite, we obtain new invariants for the data, which measure how far the approximate symmetry is from a symmetry. This will be the case for the approximate boosts. However, it turns out that for the approximate timelike translation, J diverges. This is because the approximate timelike translation grows like r at infinity and not like a constant as the timelike translation. The new invariant (which will be denoted by λ) is precisely the coefficient of r in this expansion.

The variational principle (7) is a generalization (which includes the lapse function η and it is of fourth order) of a variational principle for X^a studied in [8], which is known as the minimal distortion gauge (see also the interesting discussion in [9]). Finally, we want to point out that the operators $D\Phi$ and $D\Phi^*$ have been recently used to construct new kinds of solutions for the constraint equations [10–13].

The time-symmetric case.—If $K_{ab} = 0$ and $X_a = o(1)$ then, contracting with (η, X^a) Eq. (6), and integrating by parts, one concludes that $D_{(a}X_{b)} = 0$. Hence X^a is a Killing vector of h_{ab}. But since there are no Killing vectors which go to zero at infinity (see [14]), we have $X^a = 0$. Then, in this case, the operators (4), (5) reduce to

$$\mathcal{P}(\gamma_{ab}) = D_a D_b \gamma^{ab} - R_{ab}\gamma^{ab} - \Delta\gamma, \qquad (8)$$

$$\mathcal{P}^*(\eta)_{ab} = D_a D_b \eta - \eta R_{ab} - \Delta\eta h_{ab}. \qquad (9)$$

Using that $D^a \mathcal{P}^*(\eta)_{ab} = -\eta D_b R/2$ and $h^{ab}\mathcal{P}^*(\eta)_{ab} = -2\Delta\eta - \eta R$ we obtain

$$\mathcal{P}\mathcal{P}^*(\eta) = 2\Delta\Delta\eta - D^a D^b \eta R_{ab} + \eta R_{ab}R^{ab}$$
$$+ \frac{1}{2}\eta\Delta R + 2R\Delta\eta + \frac{3}{2}D^a\eta D_a R \qquad (10)$$

When $R = 0$ the last three terms in (10) vanish and the operator $\mathcal{P}\mathcal{P}^*$ has a very simple expression. For an arbitrary domain $\Omega \subset S$ the boundary term is given by

$$\int_\Omega \gamma^{ab}\mathcal{P}^*(\eta)_{ab}\, d\mu = \int_\Omega \mathcal{P}(\gamma_{ab})\eta\, d\mu$$
$$+ \oint_{\partial\Omega} (\eta D_a\gamma - \gamma D_a\eta$$
$$+ \gamma_{ab}D^b\eta - \eta D^b\gamma_{ab})n^a\, ds, \qquad (11)$$

where ds is the surface element with respect to h_{ab}, and n^a is the unit normal of $\partial\Omega$ pointing in the outward direction.

The falloff conditions on the fields can be conveniently written in terms of weighted Sobolev spaces H^s_β, where s is a non-negative integer and β is a real number (for [15–17] and references therein, we will use the index notation of [15]). We say that $\eta \in H^\infty_\beta$ if $\eta \in H^s_\beta$ for all s. The functions in H^∞_β are smooth in $\mathbb{R}^3$ and have the falloff at infinity $\partial^l\eta = o(r^{\beta - |l|})$.

For simplicity, we have not included matter fields in the constraint map (1). But then, if the metric is static and the topology of the data is $\mathbb{R}^3$ it should be flat. In order to have nonflat, vacuum, static metrics (i.e., Schwarzschild) we will allow S to have n *asymptotic ends*, that is, for some compact set Ω we have that $S \setminus \Omega = \sum_{k=1}^n S_k$, where S_k are open sets diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$. Each set S_k is called an *end*. We will also assume that (S, h_{ab}) is *asymptotically flat*: at each end, S_k, there exists a coordinate system $x^j_{(k)}$ such that we have in these coordinates

$$h_{ij} - \delta_{ij} \in H^\infty_\tau, \quad \tau \leq -1/2 \qquad (12)$$

where δ_{ij} denotes the flat metric and $i, j \cdots$, which take values 1, 2, 3 denote the coordinates indices. We will smoothly extend the coordinate system $x^j_{(k)}$ to be zero in $S \setminus S_k$.

We say that β is *exceptional* if β is a nonpositive integer, and we say that β is *nonexceptional* if it is not exceptional. The manifold (S, h_{ab}) will be called *static* if there exists a function $\nu \in H^4_{1/2}$ such that $\mathcal{P}^*(\nu)_{ab} = 0$. Note that in this definition we allow ν to have zeros (horizons) on S, as, for example, in Schwarzschild initial data.

We are interested in the kernel of $\mathcal{P}\mathcal{P}^*$ in H^4_β. Define

$$N(\beta) = \dim\ker(\mathcal{P}\mathcal{P}^*:H^4_\beta \to H^0_{\beta-4}). \qquad (13)$$

The main result is given by the following theorem.

Theorem 1: *Let (S, h_{ab}) be a complete, smooth, asymptotically flat, Riemannian manifold, with n asymptotic ends.* Assume that $\eta \in H^4_\beta$ satisfies

$$\mathcal{P}\mathcal{P}^*\eta = 0. \qquad (14)$$

Then, $\eta \in H^\infty_\beta$. Moreover, assume that β is nonexceptional, then, we have the following:

(i) If $\beta \leq 0$ then $N(\beta) = 0$.

(ii) If $0 < \beta \leq 1/2$ then $N(\beta) \leq 1$, and $N(\beta) = 1$ if and only if (S, h_{ab}) is static. In this case we have $R = 0$ and the unique static solution ν, $\mathcal{P}^*\nu = 0$, has at each end, S_k, the following falloff

$$\nu - \nu_0^{(k)} = o(r^\tau) \quad \partial^l \nu = o(r^{\tau - |l|}) \qquad (15)$$

where $\nu_0^{(k)} \neq 0$ is a constant and τ is given by (12).

(iii) If $1 < \beta < 2$ then $N(\beta) \geq 4n$. At each end we have the following four linear independent solutions of Eq. (14)

$$\alpha_{(k)} = \lambda_{(k)} r_{(k)} + \hat{\alpha}_{(k)}, \qquad \eta^j_{(k)} = x^j_{(k)} + \hat{\eta}^j_{(k)} \qquad (16)$$

with $\hat{\alpha}_{(k)}, \hat{\eta}^j_{(k)} \in H^4_{1/2}$. Where $\lambda_{(k)}$ are constant and $\lambda_{(k)} = 0$ for some k if and only if (S, h_{ab}) is static.

(iv) In the particular case $S = \mathbb{R}^3$ (which implies $n = 1$), we obtain more information:

(ii') If $0 < \beta < 1$ then $N(\beta) \leq 1$ and $N(\beta) = 1$ if and only if (S, h_{ab}) is static and hence flat.

(iii') If $1 < \beta < 2$ then $N(\beta) = 4$.

Proof: The differential operator $\mathcal{P}\mathcal{P}^*$ is an elliptic operator of fourth order with smooth coefficients. Using the decay assumption on the metric (12) one can easily check that $\mathcal{P}\mathcal{P}^*$ is asymptotically homogeneous of degree m with $m \geq 4$ (for the definition of this concept see for example [16]—this is the standard assumption on the coefficients for elliptic operator on weighted Sobolev spaces—see also [17]). Then, by the weighted Sobolev estimate [17], it follows that $\eta \in H^s_\beta$ for every s.

(i)–(ii). We use Eq. (11) with $\gamma^{ab} = \mathcal{P}^*(\eta)^{ab}$ and $\Omega = S$ to obtain

$$\int_S \mathcal{P}^*(\eta)^{ab}\mathcal{P}^*(\eta)_{ab} d\mu_h = \oint_{\partial S} B_a n^a ds, \qquad (17)$$

with

$$B_a = -2\eta D_a \Delta \eta + 2\Delta \eta D_a \eta + \mathcal{P}^*(\eta)_{ab} D^b \eta - \frac{1}{2}\eta^2 D_a R, \qquad (18)$$

where the boundary integral is performed in a two sphere at infinity at every end. Since $\partial^l \eta = o(r^{\beta - |l|})$ we obtain $B = o(r^{2\beta - 3})$. If $\beta \leq 1/2$ then $2\beta - 3 \leq -2$ and the boundary term vanished because $ds = O(r^2)$. We conclude that for $\beta \leq 1/2$ we have $\mathcal{P}\mathcal{P}^*(\eta) = 0 \Leftrightarrow \mathcal{P}^*(\eta) = 0$. That is, for $\beta \leq 1/2$ the kernel is not trivial if and only if the metric is static. To prove (i) we use the result that there exists no spacetime Killing vectors which go to zero at infinity [14,18]. The falloff behavior for Killing vectors (15) was proved in [18]. Because the constants $\nu_0^{(k)}$ are

nonzero, this falloff implies that ν is unique: assume that there exists another solution ν', rescale ν' such that $\nu' - \nu = o(r^\tau)$ at some end and this contradicts (15). Finally, we prove that static implies $R = 0$: in [11] it has been proved that static implies that R is constant; by our falloff assumption it should be zero.

(iii) We use the Fredholm alternative in weighted Sobolev spaces (see [16,19]; note that we use different conventions for the weights) to prove the existence of the four independent solutions at each end: the equation $\mathcal{P}\mathcal{P}^*\eta = F$, with $F \in H^0_{\beta - 4}$ will have a solution $\eta \in H^4_\beta$ if and only if

$$\int_S F\nu d\mu = 0 \qquad (19)$$

for all $\nu \in H^0_{\beta'}$ such that $\mathcal{P}\mathcal{P}^*\nu = 0$, with $\beta' = 1 - \beta$.

We prove first the existence of η^j (in the following we will suppress the end label (k); all the calculations are done in one arbitrary end). Set $\eta^j = x^j + \hat{\eta}^j$, then $\hat{\eta}^j$ satisfies the equation

$$\mathcal{P}\mathcal{P}^*(\hat{\eta}^j) = -\mathcal{P}\mathcal{P}^*(x^j). \qquad (20)$$

We have that $\mathcal{P}\mathcal{P}^*(x^k) \in H^0_{\tau - 3}$. Since $\tau \leq -1/2$ we can take $\hat{\eta}^j \in H^4_{1/2}$ in Eq. (20). From the discussion above we have that a solution $\hat{\eta}^j \in H^4_{1/2}$ of (20) will exist if and only if the right hand side satisfies the condition (19). Since $\beta = \beta' = 1/2$ in this case, we can use (ii) to conclude that a nontrivial ν will exist if and only if (S, h_{ab}) is static. Then, if the metric is not static we do not have any restrictions and the solutions $\hat{\eta}^j$ exist. If the metric is static, we compute

$$\int_S \nu \mathcal{P}\mathcal{P}^*(x^j) = \int_S x^j \mathcal{P}\mathcal{P}^*(\nu) + \oint_{\partial S} B_a n^a ds, \qquad (21)$$

$$= \oint_{\partial S} B_a n^a ds, \qquad (22)$$

where

$$B_a = -2\nu D_a \Delta x^j + 2\Delta \nu D_a x^j + \mathcal{P}^*(x^j)_{ab} D^b \nu$$
$$- 2x^j D_a \Delta \nu + 2\Delta x^j D_a \nu, \qquad (23)$$

and we have used that $\mathcal{P}^*(\nu)_{ab} = 0$. We use the falloff (15) for ν, the falloff (12) for the metric, and the fact that $\Delta_\delta x^j = 0$ (where Δ_δ is the flat Laplacian), to conclude that $B_a = o(r^{-2})$; then the boundary integral (22) vanishes and the solution exists also when the metric is static.

For the solution α we proceed in an analogous way. Let $\lambda \neq 0$ be an arbitrary constant. We have that $\mathcal{P}\mathcal{P}^*(\lambda r) \in H^0_{\tau - 3}$ (here we use that $\Delta_\delta \Delta_\delta r = 0$). If the metric is not static there exists a solution $\hat{\alpha} \in H^4_{1/2}$ of

$$\mathcal{P}\mathcal{P}^*(\hat{\alpha}) = -\mathcal{P}\mathcal{P}^*(\lambda r). \qquad (24)$$

If the metric is static we can compute condition (19) as we did in Eqs. (21)–(23)

$$\int_S \nu \mathcal{P}\mathcal{P}^*(\lambda r) = -2 \oint_{\partial S} \nu D_a \Delta(\lambda r) n^a ds \qquad (25)$$

$$= 16\pi\lambda\nu_0, \tag{26}$$

where $\nu_0 \neq 0$ is the constant given in (15) and we have used $\Delta_\delta r = 2/r$. Then, the constant λ is zero if and only if the metric is static.

For every end, we have constructed four independent solutions, then $N(\beta) \geq 4n$.

(iv) The Fredholm index of $\mathcal{P}\mathcal{P}^*$ is given by

$$\iota(\beta) = N(\beta) - N(1 - \beta). \tag{27}$$

When $S = \mathbb{R}^3$ the index $\iota(\beta)$ of $\mathcal{P}\mathcal{P}^*$ is equal to the index $\iota_0(\beta)$ of the flat operator $\Delta_\delta\Delta_\delta$ (see [19]). To prove (ii')–(iii') we will calculate $\iota_0(\beta)$ and use $\iota(\beta) = \iota_0(\beta)$.

Assume $\Delta_\delta\Delta_\delta\nu = 0$, that is $\Delta_\delta w = 0$, $w = \Delta_\delta\nu$. If $\nu \in H^k_\beta$ with $\beta < 2$, then $w \in H^{k-2}_{\beta-2}$, we use the Liouville theorem proved in Corollary 1.9 of [15] to conclude that $w = 0$. Then $\Delta_\delta\nu = 0$, using again this result we conclude that for $0 < \beta < 1$ we have $\nu = 1$, and for $1 < \beta < 2$ we have $\nu = 1, x^j$. Then we have $\iota_0(\beta) = 0$ for $0 < \beta < 1$ and $\iota_0(\beta) = 4$ for $1 < \beta < 2$.

(ii') We have $0 = i_0(\beta) = i(\beta) = N(\beta) - N(\beta')$. We only need to prove the case $1/2 < \beta < 1$. In this case we have $0 < \beta' < 1/2$, using (ii) we get $N(\beta') \leq 1$, and the equality holds if and only if the metric is static.

(iii') We have $4 = i_0(\beta) = i(\beta) = N(\beta) - N(\beta')$, and $\beta' < 0$; using (i) we get $N(\beta') = 0$.

The new invariants are given by the constants $\lambda_{(k)}$. They can be written as a boundary integral at each end S_k

$$\lambda_{(k)} = \frac{-1}{8\pi} \oint_{\partial S_k} n^a D_a \Delta\alpha_{(k)} ds \tag{28}$$

Note that $\alpha_{(k')} = o(r^{1/2})$ at $S_{k'}$, for $k' \neq k$, then if we calculate (28) for $\alpha_{(k')}$ we get zero. Also, if we compute this boundary for η^j we get zero.

If we assume $R = 0$ (in the previous theorem the metric is not assumed to satisfy the constraint equations), using Eq. (10) we get another representation for $\lambda_{(k)}$ as a volume integral

$$\lambda_{(k)} = \frac{1}{16\pi} \int_S \alpha_{(k)} R_{ab} R^{ab} d\mu. \tag{29}$$

Since we have the freedom to rescale α by an arbitrary constant, we need to normalize α if we want to compare λ for different data. In the case of $S = \mathbb{R}^3$ we have a natural normalization. Let h_{ab} be a metric on $\mathbb{R}^3$. Assume that there exists a smooth family $h_{ab}(\epsilon)$ such that for $0 \leq \epsilon \leq 1$ it satisfies the hypothesis of the previous theorem and $h_{ab}(1) = h_{ab}$ and $h_{ab}(0) = \delta_{ab}$. Then, for every ϵ we get a solution $\alpha(\epsilon)$. The flat solutions $\alpha(0)$ are the constants. We normalize α, setting $\alpha(0) = 1$. With this normalization, we can calculate λ up to second order in ϵ

$$\lambda \approx \frac{\epsilon^2}{16\pi} \int_S \dot{R}^{ab} \dot{R}_{ab} d\mu + O(\epsilon^3) \tag{30}$$

where $\dot{R}^{ab} = dR_{ab}(\epsilon)/d\epsilon|_{\epsilon=0}$.

The solutions η^j provide a coordinate system near infinity. In the nonflat case this system is unique up to rotations, the translation freedom at infinity is fixed because the constants are not solutions of Eq. (14).

Since $\mathcal{P}^*(\eta^j) \in H^0_{-3/2}$ we have that the functional $J(\eta^j)$ defined in (7) is finite. The solutions η^j are the minimum of this functional with the boundary conditions $\eta^j = x^j + o(r^{1/2})$ at infinity. The numbers $J(\eta^j)$ are a measure of how far the metric h_{ab} is of having a boost Killing vector. The falloff conditions of η^j are like the ones for the boost Killing vectors. In contrast, the solution α has a different falloff as the time translation. This difference is reflected also in the fact that $J(\alpha)$ is infinite, it grows like $J(\alpha) \approx \lambda r$ at infinity.

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