Mirror Inversion of Quantum States in Linear Registers

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Transfer of data in linear quantum registers can be significantly simplified with preengineered but not dynamically controlled interqubit couplings. We show how to implement a mirror inversion of the state of the register in each excitation subspace with respect to the center of the register. Our construction is especially appealing as it requires no dynamical control over individual interqubit interactions. If, however, individual control of the interactions is available then the mirror inversion operation can be performed on any substring of qubits in the register. In this case, a sequence of mirror inversions can generate any permutation of a quantum state of the involved qubits.

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The network (circuit) model of quantum computation is justifiably the most popular model for investigating both computational power and possible experimental realizations of quantum computers. One of its many appealing features is the reduction of quantum computation to prescribed sequences of elementary operations (quantum logic gates) performed either on individual qubits or on pairs of qubits [1]. However, a tacit assumption that single- and two-qubit operations are much easier to implement than multiqubit operations is not always valid. In fact, there are potentially interesting technologies, for example, optical lattices [2], arrays of quantum dots [3–6], or NMR [7,8], in which joint operations on several qubits are relatively easy whereas addressing individual qubits poses a substantial experimental challenge. Thus it is important to investigate quantum computation with limited control over individual qubits. Here we show that transfer of data in quantum registers can be significantly simplified with preengineered but not dynamically controlled interqubit couplings.

It is known that quantum computation could in principle be performed by a chain of qubits coupled via the Heisenberg or the XY interactions [9], and that it suffices to control the qubits collectively [10]. Such a chain of qubits represents a quantum register. Further simplifications to this model have been recently introduced by Zhou *et al.* [11] and by Benjamin and Bose [12]. Still, a significant number of elementary operations in the process of computation is delegated to moving around quantum states of individual qubits. We show how to simplify these operations by implementing a mirror inversion of a quantum state with respect to the center of the chain. More precisely, given a chain of $N+1$ qubits described by the wave function $\Psi(s_0, \ldots s_N)$, where $s_n = 0, 1$ denotes the bit values of the nth qubit, we show how to implement the transformation R

$$R\Psi(s_0, s_1 \ldots, s_{N-1}, s_N) = (\pm)\Psi(s_N, s_{N-1}, \ldots, s_1, s_0). \tag{1}$$

Our construction has the advantage that it can be done without applying any dynamical control to the qubits; it only exploits the natural dynamics of the chain governed by a preengineered mirror-periodic Hamiltonian H such that $\exp(-iTH) = R$ for some time T.

Apart from obvious applications, such as a perfect quantum wire or a "data bus" linking the two opposite ends of the chain, the study of periodic and mirror-periodic dynamics of chains of spins with nonhomogenous couplings is an interesting subject on its own, with potential applications outside quantum computation, e.g., in the design of frequency standards and in mathematical finance.

Consider $N+1$ interacting qubits, or spin-1/2 particles, in a quantum register. We choose the Hamiltonian of the system to be of the XY type

$$H = \frac{1}{2}\sum_{\ell=0}^{N-1} J_\ell(\sigma_\ell^x \cdot \sigma_{\ell+1}^x + \sigma_\ell^y \cdot \sigma_{\ell+1}^y) - \frac{1}{2}\sum_{\ell=0}^{N} h_\ell(\sigma_\ell^z - 1), \tag{2}$$

where J_ℓ is the coupling strength between the qubits located at sites ℓ and $\ell+1$, and h_ℓ is the "Zeeman" energy of a qubit at site ℓ. Please note that here, ℓ labels the position of a qubit in the register, whereas the three Pauli matrices are denoted as σ^x, σ^y, and σ^z.

Now our task is to find the values J_ℓ and h_ℓ for which the Hamiltonian H is mirror-periodic. The total z component of the spin, given by

$$\sigma_{\text{tot}}^z := \sum_{\ell=0}^{N} \sigma_\ell^z \tag{3}$$

is conserved, i.e., $[\sigma_{\text{tot}}^z, H] = 0$. Hence the Hilbert space of the register decomposes into invariant subspaces, each of which is a distinct eigenspace of the operator $\sigma^z_{\rm tot}$. The eigenspace with eigenvalue $(2M - N - 1)/2$ corresponds to exactly M qubits having bit value 1. Let us denote this subspace by S_M.

For convenience of our exposition, we adopt here the standard fermionization technique [13]. We will view the register as a lattice with $N + 1$ sites, some of which are occupied by indistinguishable and noninteracting, spinless fermions. The bit values 1 and 0 indicate the presence and the absence of the fermion at a given lattice site and the Pauli exclusion principle prevents two or more fermions to occupy the same site. The subspace S_M corresponds to the M-fermion sector, in which M of the $N + 1$ lattice sites are occupied by fermions. The Jordan-Wigner transformation

$$a_\ell = \left(\prod_{k<\ell}\sigma^z_k\right)\frac{\sigma^x_\ell + i\sigma^y_\ell}{2}, \qquad a^\dagger_\ell = \left(\prod_{k<\ell}\sigma^z_k\right)\frac{\sigma^x_\ell - i\sigma^y_\ell}{2} \tag{4}$$

allows to rewrite the Hamiltonian (2) in the second quantization form using the fermionic operators a_ℓ and $a^\dagger_\ell$,

$$H = \sum_{\ell=0}^{N-1} J_\ell(a^\dagger_\ell a_{\ell+1} + a^\dagger_{\ell+1}a_\ell) + \sum_{\ell=0}^{N} h_\ell a^\dagger_\ell a_\ell. \tag{5}$$

The Hamiltonian H in (5) describes a set of $N + 1$ noninteracting (or free) fermions which hop between adjacent sites of the lattice and are subject to a nonuniform magnetic field, denoted by h_ℓ, $\ell = 0, 1, \ldots, N$. Let $|\ell\rangle$ denote a state in which there is a single fermion at the site ℓ and all other sites are empty. Then the set of states $\{|\ell\rangle\}$ forms a basis spanning the subspace S_1. In this single-particle basis, the Hamiltonian H is represented by the matrix

$$\begin{pmatrix} h_0 & J_0 & 0 & \cdots & 0 \\ J_0 & h_1 & J_1 & \cdots & 0 \\ 0 & J_1 & h_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_{N-1} \\ 0 & 0 & 0 & J_{N-1} & h_N \end{pmatrix}. \tag{6}$$

The dynamics of the register is completely determined by the eigenvalues and eigenvectors of the above matrix. Let us denote the energy eigenvalues of the matrix by E_k, where $k = 0, 1, \ldots, N$, and the corresponding energy eigenfunction by $\phi_k(\ell)$ (where $\ell \in \{0, 1, \ldots, N\}$). The latter corresponds to a single fermion at the site ℓ of the chain. In the M-fermion sector, the energy of M fermions occupying orbitals $0 \le k_1 < \cdots < k_M \le N$ is then given by

$$E_{k_1,\ldots,k_M} = \sum_{i=1}^{M} E_{k_i} \tag{7}$$

and the corresponding M-particle energy eigenfunction can be written as the Slater determinant

$$\Phi_{k_1,\ldots,k_M}(\ell_1,\ldots,\ell_M) = \frac{1}{\sqrt{M!}}\begin{vmatrix} \phi_{k_1} & (\ell_1) & \cdots & \phi_{k_1} & (\ell_M) \\ \vdots & & \ddots & & \vdots \\ \phi_{k_M} & (\ell_1) & \cdots & \phi_{k_M} & (\ell_M) \end{vmatrix}. \tag{8}$$

The eigenfunction $\Phi_{k_1,\ldots,k_M}(\ell_1,\ldots,\ell_M)$ is completely antisymmetric. Let us now see how this eigenfunction is related to the wave function of the quantum register.

In the subspace S_M, the wave function of the register $\Psi(s_0,\ldots,s_N)$ can be expressed as $\Psi(\ell_1,\ldots,\ell_M)$, where $\ell_1,\ldots,\ell_M$ label the qubits that have bit values equal to 1. Each of the remaining qubits have bit value 0. In other words, the value of $\Psi(\ell_1,\ldots,\ell_M)$ gives the probability amplitude that the qubits located at the sites $\ell_1, \ell_2, \ldots, \ell_M$ represent binary "1" and all other qubits represent binary "0." Note that the wave function $\Psi(\ell_1,\ldots,\ell_M)$ is symmetric under an interchange of its labels, and is hence bosonic. It can, however, be expressed in terms of the fermionic wave functions $\Phi_{k_1,\ldots,k_M}(\ell_1,\ldots,\ell_M)$ in the following manner: in the sector $\ell_1 < \ell_2 < \ldots < \ell_M$, the wave function of the register corresponding to the energy eigenvalue $E_{k_1,\ldots,k_M}$ is set equal to the fermionic eigenfunction $\Phi_{k_1,\ldots,k_M}(\ell_1,\ldots,\ell_M)$:

$$\Psi(\ell_1,\ldots,\ell_M) \equiv \Psi_{k_1,\ldots,k_M}(\ell_1,\ldots,\ell_M) \tag{9}$$

$$= \Phi_{k_1,\ldots,k_M}(\ell_1,\ldots,\ell_M). \tag{10}$$

In the other sectors, the two differ by the sign giving the parity of the permutation required to reshuffle the arguments in increasing order.

A Hamiltonian is said to be *mirror-periodic* if it satisfies

$$e^{-iTH}\Psi(\ell_1,\ldots,\ell_M) = (\pm 1)\Psi(N-\ell_1,\ldots,N-\ell_M) \tag{11}$$

for each $1 \le M \le N$, the sign depending on M and N only. Since $\Psi(\ell_1,\ldots,\ell_M)$ is bosonic, we can choose $\ell_1 < \ell_2 < \ldots < \ell_M$. Now,

$$e^{-iTH}\Psi(\ell_1,\ldots,\ell_M) = e^{-iTH}\Phi_{k_1,\ldots,k_M}(\ell_1,\ldots,\ell_M). \tag{12}$$

Our aim is to find Hamiltonians H for which the right-hand side of Eq. (12) is given by $\Phi_{k_1,\ldots,k_M}(N-\ell_1,\ldots,N-\ell_M)$. This would imply mirror periodicity of H since

$$\Phi_{k_1,\ldots,k_M}(N-\ell_1,\ldots,N-\ell_M) = (-1)^{[M(M-1)]/2}\Psi(N-\ell_1,\ldots,N-\ell_M) \tag{13}$$

by the above discussion.

The mirror periodicity with period T implies periodicity with period $2T$, which in turn implies that for all k the quantity $2TE_k$ is an integer multiple of 2π in units for which $\hbar = 1$ and $\phi_k(N-\ell) = \pm\phi_k(\ell)$.

We found two families of mirror-periodic Hamiltonians: one (A) with linear spectrum and the other (B) with quadratic spectrum. An alternative proof of mirror periodicity for the case (A), in the single-particle sector, was given by Christandl *et al.* [14]. The proof relied on identifying the Hamiltonian operator with the generator of space rotations and employed group theoretical methods. In this Letter we recognize that the mirror periodicity extends to all multiparticle sectors and that it is also shared by another finite quantum chain with eigenfunctions given by Hahn polynomials. Let us now discuss cases (A) and (B) in detail:

(A) The quantum chain with linear spectrum $P(k) = k$ has eigenfunctions $\phi_k(\ell)$ proportional to Krawtchouk polynomials. This polynomial basis has been used by Atakishiev *et al.* [15] to construct finite quantum chains admitting periodic solutions.

The Krawtchouk quantum chain, which is mirror-periodic of period $T = \pi$, has couplings

$$J_\ell = \sqrt{(\ell + 1)(N - \ell)}; \qquad h_\ell = 0. \tag{14}$$

The Krawtchouk polynomials are defined in terms of the hypergeometric functions F as

$$K_k(\ell, p, N) = {}_2F_1\left(\begin{matrix} -k, -\ell \\ -N \end{matrix} \middle| \frac{1}{p} 1\right), \tag{15}$$

where $k = 0, 1, 2, \ldots, N$. The energy eigenfunctions are

$$\phi_k(\ell) = c_k \sqrt{w(\ell)} K_k\left(\ell, \frac{1}{2}, N\right), \tag{16}$$

where c_k and $w(\ell)$ are given by

$$c_k = \sqrt{\frac{(-N)_k}{(-1)^k k!}}, \qquad w(\ell) = \frac{1}{2^N}\binom{N}{\ell}. \tag{17}$$

The corresponding eigenvalues are $E_k = -k$. In the definitions above we have used the Pochhammer symbol, $(N)_k$ defined as

$$(N)_k = N(N + 1)\ldots(N + k - 1), \qquad k = 1, 2, 3, \ldots \tag{18}$$

with $(N)_0 = 1$, and the generalized binomial symbol expressed in terms of the Γ function as

$$\binom{N}{\ell} = \frac{\Gamma(N + 1)}{\Gamma(N - \ell + 1)\Gamma(\ell + 1)}. \tag{19}$$

For a more comprehensive description of the Krawtchouk polynomials, we refer to [16].

The energy eigenfunctions satisfy the property of reflection symmetry (or antisymmetry):

$$\phi_k(N - \ell) = (-1)^k \phi_k(\ell) \tag{20}$$

for all ℓ and all $k = 0, 1, 2, \ldots, N$. This follows from the following property of the Krawtchouk polynomials:

$$K_k\left(N - \ell; \frac{1}{2}, N\right) = (-1)^k K_k\left(\ell; \frac{1}{2}, N\right), \tag{21}$$

and the fact that the weight function in (17) is symmetric. The phases $(-1)^k$ in (20) perfectly offset the dynamical phases acquired after a time period $T = \pi$, as $\exp(-iTE_k) = (-1)^k$. This shows that the chain defined by the Hamiltonian corresponding to the Krawtchouk polynomials is mirror symmetric with period π.

The dynamics of the Krawtchouk quantum chain of $N + 1$ sites is the same as that of a spin $s = N/2$ particle governed by the Hamiltonian $H_s = 2\mathbf{s}_x$. This Hamiltonian acts as follows on the basis vectors $|m\rangle, m = -s, \ldots, s$:

$$H_s|m\rangle = R(m)|m - 1\rangle + L(m)|m + 1\rangle,$$

where

$$R(m) = \sqrt{s(s + 1) - m(m - 1)}$$
$$L(m) = \sqrt{s(s + 1) - m(m + 1)}.$$

It is possible to establish a relation between H_s and a mirror-periodic Krawtchouk chain of $N + 1 = 2s + 1$ sites. This is done by identifying the state $|\ell\rangle$ corresponding to a single-particle occupying the site ℓ with the state $|m\rangle$, where $m = s - \ell$. In this case, $R(m)$ reduces to J_ℓ and $L(m)$ reduces to $J_{\ell-1}$, showing that the spin Hamiltonian H_s is equivalent to the mirror-periodic Krawtchouk Hamiltonian.

(B) One can use Hahn polynomials to find a family of mirror-periodic quantum chains whose period is an integer multiple of π with quadratic spectrum $E_k = k(k + 2\alpha + 1)$, where α is of the form

$$\alpha = \frac{2p + 1}{2q} \tag{22}$$

where p, q are integers with $q \neq 0$. The couplings are

$$J_\ell = \sqrt{(\ell + 1)(N - \ell)(\alpha + N - \ell)(\alpha + \ell + 1)} \tag{23}$$

and the Zeeman terms are given by

$$h_\ell = \frac{N^2}{2} + (\alpha + 1)N - 2\left(\ell - \frac{N}{2}\right)^2. \tag{24}$$

This model has eigenfunctions $\phi_k(\ell)$ given by Hahn polynomials. The Hahn polynomials are defined in terms of the hypergeometric functions F as

$$Q_k(\ell; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -k, k + \alpha + \beta + 1, \ell \\ \alpha + 1, -N \end{matrix} \middle| 1\right), \tag{25}$$

where $k = 0, 1, 2, \ldots, N$. The energy eigenfunctions of the Hamiltonian (2) are given by

$$\phi_k(\ell) = c_k \sqrt{w(\ell)} Q_k(\ell; \alpha, \alpha, N), \tag{26}$$

where c_k is the constant

$$c_k = \sqrt{\frac{(2k + 2\alpha + 1)(N!)^2}{(k + 2\alpha + 1)_{N+1} k!(N - k)!}}, \tag{27}$$

and $w(\ell)$ is the weight function

$$w(\ell) = \binom{\alpha + \ell}{\ell}\binom{\alpha + N - \ell}{N - \ell}. \tag{28}$$

For further details on Hahn polynomials, see [16].

To show that the $\phi_k(\ell)$ are either reflection symmetric or antisymmetric, we notice that

$$Q_k(N - \ell; \alpha, \alpha, N) = (-1)^k Q_k(\ell; \alpha, \alpha, N), \tag{29}$$

and that the weight function in (28) is symmetric. Hence, $\phi_k(N - \ell) = (-1)^k \phi_k(\ell)$ for all ℓ and all $k = 0, 1, 2, \ldots, N$. If α satisfies (22) and $T = q\pi$, the phases $(-1)^k$ perfectly offset the dynamical phases. In fact,

$$\exp(-iTE_k) = \exp(-i\pi[q(k^2 + k) + (2p + 1)k]) = (-1)^k, \tag{30}$$

since $k^2 + k$ is even for all $k = 0, \ldots, N$. This shows that the Hahn chain is mirror-periodic of period $T = q\pi$.

The Hahn chain Hamiltonian in the special case $q = 1$, i.e., when $\alpha = (2p + 1)/2$ is half-integer, is related to atomic Hamiltonians with $\mathbf{L} \cdot \mathbf{S}$ coupling. Consider the Hamiltonian

$$H_{LS} = \mathbf{L} \cdot \mathbf{S} \tag{31}$$

restricted to the sector with fixed total angular momentum L, total spin S, and with projections along a fixed axis adding up to zero, i.e., $M = M_L + M_S = 0$. The Hamiltonian in (31) acts as follows on the basis vectors $|M_S\rangle \equiv |L, S; M_L, M_S\rangle$:

$$H_{LS}|M_S\rangle = D|M_S\rangle + R|M_S - 1\rangle + L|M_S + 1\rangle, \tag{32}$$

where

$$D \equiv D(M_S) = -M_S^2 \tag{33}$$

$$R \equiv R(M_S) = \frac{1}{2}\sqrt{(L + M_S)(L - M_S + 1)} \times \sqrt{(S + M_S)(S - M_S + 1)} \tag{34}$$

$$L \equiv L(M_S) = \frac{1}{2}\sqrt{(L - M_S)(L + M_S + 1)} \times \sqrt{(S - M_S)(S + M_S + 1)}. \tag{35}$$

Assuming that $S < L$ and that S is a half-integer, it is possible to establish a relation between H_{LS} and a mirror-periodic Hahn chain of $N = 2S$ sites and $\alpha = L - S$. This is done by identifying the state $|\ell\rangle$ corresponding to a single particle occupying the site ℓ with the state $|M_S\rangle$, where $M_S = S - \ell$. We find $R(M_S) = \frac{1}{2}J_\ell$, $L(M_S) = \frac{1}{2}J_{\ell-1}$, and $D(M_S) = \frac{1}{2}h_\ell +$ const. This shows that the LS coupling Hamiltonian is proportional to a mirror-periodic Hahn Hamiltonian up to a constant energy shift.

In conclusion, in this Letter we have demonstrated how to simplify transfer of data in quantum registers by implementing a mirror inversion of a quantum state with respect to the center of the register. Our construction is especially appealing as it requires no dynamical control over individual qubits but only preengineered inter-qubit couplings. If, however, individual control of the interactions is available, then the mirror inversion operation can be performed on any substring of qubits in the register. In this case, a sequence of mirror inversions can generate any permutation of a quantum state of the involved qubits.

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