Decoherence and Multipartite Entanglement

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We study the dynamics of multipartite entanglement under the influence of decoherence. A suitable generalization of concurrence reveals distinct scaling of the entanglement decay rate of Greenberger-Horne-Zeilinger and *W* states, for various environments.

The notion of entangled states is, since the early days of quantum mechanics, a key concept when it comes to distinguish between the quantum and the classical worlds. Besides this fundamental aspect, entanglement attracts considerable interest since it can be viewed as an indispensable ingredient for quantum information processing. Numerous experiments have recently been carried out in this area, in particular, on the controlled generation of entanglement between many quantum systems [1–5]—such as to accomplish fundamental scalability requirements for quantum computation.

An obstacle for the production and observation of multipartite entanglement resides in its fragility under the unavoidable interaction with the environment. This challenges the experimentalist, but also demands a proper theoretical description of multipartite entanglement in open systems. Despite recent progress in the understanding of decoherence processes in quantum systems, a systematic characterization of the environment induced loss of many-particle quantum correlations is still lacking, and this is specifically due to the difficulties in quantifying multipartite entanglement.

Even for bipartite mixed states, apart from the particular case of two-level systems where an exact solution is known [6], the situation is far from being simple. Some of the widely used indicators of entanglement, such as the positive partial transpose and negativity [7], fail to detect certain entangled states, while some other entanglement measures for mixed states require a high dimensional optimization procedure which provides only an upper bound, unable to reliably distinguish entangled from separable states. Only recently [8], a lower bound (together with a numerically manageable upper bound) for the concurrence of mixed bipartite quantum states was derived. In the multipartite case, one usually has to deal with bipartite cuts, where the N individual constituents are partitioned into two (arbitrarily chosen) subgroups. The available entanglement measures for bipartite systems thus become applicable, though different cuts or partitions may lead to different values of entanglement and, furthermore, the number of possible bipartite cuts increases rapidly with N.

To improve on this situation, we scrutinize here the effects of decoherence on multipartite entanglement measured by a suitable generalization of concurrence—sensitive to multipartite correlations. Moreover, since different experimental strategies to produce multipartite entanglement are subject to different sources of decoherence, we implement our approach for different types of environment coupling, acting on different types of initially maximally entangled multipartite quantum states. This finally provides a versatile toolbox to assess the scaling of entanglement decay with the system size under very general conditions.

To start with, we assume that each individual particle of the system interacts independently with the environment and, therefore, undergoes a local decoherence process which is mediated by dissipation, noise, or dephasing. Dissipative effects are described by the coupling of the system to a thermal bath at zero temperature, and can represent, e.g., the spontaneous decay of a two-level atom induced by its interaction with the vacuum modes of the ambient electromagnetic field. Noisy dynamics are related to another limit of the thermal reservoir, when temperature tends to infinity while the coupling strength goes to zero. Dephasing corresponds to a situation where no energy is exchanged with the environment, but only phase information is lost. All these processes can be described in terms of the master equation

$$\frac{d\rho}{dt} = \sum_{k=1}^{N}(\underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{k-1} \otimes \mathcal{L}_k \otimes \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{N-k})\rho, \quad (1)$$

where ρ is the reduced density operator of the system. The Lindblad operators $\mathcal{L}_k$, describing the independent interaction of each particle k with the reservoir, are assumed to be of the same form for all particles and can be written, in the weak coupling regime and in the Markovian limit, as

$$\mathcal{L}_k\rho = \sum_i \frac{\Gamma_i}{2}(2c_i\rho c_i^\dagger - c_i^\dagger c_i\rho - \rho c_i^\dagger c_i), \quad (2)$$

where c_i and Γ_i describe, respectively, the system-environment coupling operator and its strength. Further specializing to the case that each particle is a two-level system (the generic scenario in ion trap quantum information processing), the operators c_i in (2) can be written in terms of the Pauli matrices and are given by $c = \sigma_-$,

for the zero temperature reservoir, by $c_1 = \sigma_-$ and $c_2 = \sigma_+$, for the infinite temperature environment, and by $c = \sigma_+ \sigma_-$ for dephasing, with $\Gamma_i = \Gamma$ in all cases.

It is clear that, under the action of any of the above environments, any initially entangled state asymptotically evolves into a separable state. We are interested in the rate at which multipartite entanglement decays, and therefore need a suitable measure of multipartite entanglement which is evaluable for general states. For this purpose, we use a generalization of the concurrence of pure bipartite states [9], $C_2(\Psi) = \sqrt{2(\langle\Psi|\Psi\rangle^2 - \mathrm{Tr}\,\varrho_r^2)}$, where the reduced density matrix $\varrho_r = \mathrm{Tr}_p|\Psi\rangle\langle\Psi|$ is obtained as a partial trace of the bipartite state. For N-partite systems, one can construct $2^N - 2$ different reduced density matrices, half of which in general have a different mixing. A generalization of C_2 is therefore not unique, providing several inequivalent alternatives. For our present purpose, we focus on the specific form

$$C_N(\Psi) = 2^{1-(N/2)}\sqrt{(2^N - 2)\langle\Psi|\Psi\rangle^2 - \sum_\alpha \mathrm{Tr}\,\varrho_\alpha^2}, \quad (3)$$

where α labels all different reduced density matrices; i.e., there are $N!/(N-n)!\,n!$ different terms when ϱ_α is obtained by tracing over n different subsystems. C_N vanishes exactly if $|\Psi_N\rangle$ is N separable, i.e., $|\Psi_N\rangle = \bigotimes_{i=1}^N |\Phi_i\rangle$, where the $|\Phi_i\rangle$ are pure states of the individual subsystems, and C_N adopts its maximal value for Greenberger-Horne-Zeilinger (GHZ) states $\sum_i |i \ldots i\rangle/\sqrt{2}$. Hence, the crucial merit of our specific definition (3) is that C_N can account for real multipartite correlations [10]—as opposed to other multipartite generalizations of concurrence [11,12] which extract only bipartite correlations between single subsystems and the remainder. Also note that Eq. (3) satisfies another important requirement for our present purpose—it allows one to compare the degree of entanglement of multipartite systems with different numbers of constituents: Consider an N-partite state $|\Psi_N\rangle$ that factorizes into an $(N-1)$-partite state $|\Psi_{N-1}\rangle$ and a one component state $|\Phi\rangle$, $|\Psi_N\rangle = |\Psi_{N-1}\rangle \otimes |\Phi\rangle$. In this case, the N-partite concurrence $C_N(\Psi_N)$ simply reduces to the $(N-1)$-partite concurrence $C_{N-1}(\Psi_{N-1})$, and thus allows for a meaningful comparison of $N-1$ with N-particle states.

To monitor the time evolution of entanglement, we still need the generalization of Eq. (3) for mixed states, given as

$$C_N(\rho) = \inf \sum_i p_i C_N(\Psi_i), \quad (4)$$

where the infimum is to be found among all sets of probabilities p_i and pure states $|\Psi_i\rangle$, such that $\rho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$. In principle, this defines an optimization problem that is very difficult to solve exactly, though lower bounds for $C_N(\rho)$ are available [8], and can be determined purely algebraically in the regime where the

mixing of ρ is moderate [13,14]. Moreover, for states of rank two—that we encounter for specific time evolutions considered further down—Eq. (4) can be condensed into an expression equivalent to the one for bipartite two-level systems [6,8], and the required optimization can be performed exactly.

With the definition (4) at hand, we can monitor the multipartite concurrence C_N for the solutions $\rho(t)$ of Eq. (1). We consider two types of initial states: the GHZ state, $|\Psi_N\rangle_{\mathrm{GHZ}} = (|00\ldots0\rangle + |11\ldots1\rangle)/\sqrt{2}$, and the W state, $|\Psi_N\rangle_W = (|00\ldots01\rangle + |00\ldots10\rangle + \cdots + |10\ldots00\rangle)/\sqrt{N}$ —which are known to bear incompatible multipartite correlations, in the sense that they cannot be transformed into each other by local unitary transformations amended by classical communication [15]. Also note that these states have recently been produced [3,5] for $N = 3, 4$ in the laboratory, and that it is now within experimental reach to monitor the time dependence of their degree of entanglement by means of quantum state tomography.

As a first observation, illustrated in Fig. 1, we find an essentially perfect monoexponential decay of concurrence in all analyzed cases, with a decay rate γ which depends on the initial condition and environment model. Only for an infinite temperature reservoir does $C_N(t)$ vanish after a finite time t_{sep}, as illustrated in Fig. 1, consistent with the results obtained in [16,17] for a depolarizing channel. In contrast, for zero temperature as well
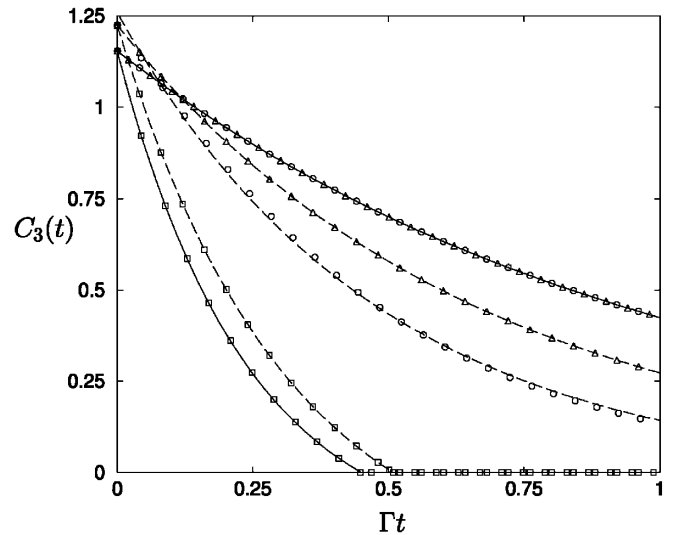

FIG. 1. Time dependence of the multipartite concurrence $C_N(t)$, for a $N = 3$-partite system initially prepared in a GHZ (dashed lines) or a W (solid lines) state. Only for an infinite temperature reservoir (squares) does the state turn separable after a finite time t_{sep}. In contrast, the zero temperature (circles) as well as the dephasing environment (triangles) induce separability only in the limit $t \to \infty$. In all cases, the numerical results are very well fitted by an exponential (solid and dashed lines).

as for dephasing environments does $C_N(t)$ vanish only for $t \to \infty$. In these two cases the concurrence of the decaying W state gets larger than that of the decaying GHZ state after a short time. In contrast to this, the situation of the initially prepared states, where the concurrence of the GHZ state is larger than that of the W state, is preserved for the infinite temperature reservoir. To allow for a consistent comparison of the decay of entanglement in arbitrary environments, and, furthermore, since experimentalists seek to maximize and to preserve entanglement by preparing pure entangled states and minimizing decoherence, rather than to await its fading out, we therefore choose γ rather than t_{sep} as the figure of merit in our analysis of entanglement decay for variable system size.

Consistently with the experimental scenario, we assume that the initial multipartite states are pure (or quasipure; i.e., the experimental preparation of the initial quantum state succeeds with essentially perfect fidelity) [3,5]. Thus, on short time scales, where the degree of mixing of ρ is expected to be small, the use of our quasipure approximation [14] is justified. Moreover, we explicitly verified that, even for considerably mixed states, this approximation provides an excellent estimate of C_N in Eq. (4) with significantly reduced computational effort. Since we always evaluate lower bounds on concurrence, all decay rates shown hereafter present an *upper* bound of the actual rate at which multipartite entanglement is lost due to contact with the environment.

Because of their primordial experimental interest, we focus on the multipartite entanglement of GHZ and W states of variable size. Figure 2 shows the scaling of their entanglement decay rates γ with N, under decoherence induced by zero temperature, infinite temperature, and dephasing environments. We see that the GHZ state (top panel of the figure) decays into a separable state with a rate which increases linearly with N, except for the small-N behavior of γ for the zero temperature environment. Indeed, the special case of a dephasing reservoir, where the density matrix is always a mixture of two pure states, and, hence, is of rank two, can be treated analytically. In this particular case, the concurrence just follows the behavior of the two nonvanishing nondiagonal elements of ρ which uniformly decay as $e^{-N\Gamma t/2}$.

Remarkably, the situation changes quite drastically for the W states (bottom plot of Fig. 2). In this case, only the infinite temperature environment gives rise to an almost linear increase of γ with N, slightly faster than for the GHZ states. In contrast, for dephasing and zero temperature reservoirs, the decay of the concurrence is *independent* of N. Moreover, the zero temperature case once again allows for an analytic solution (as above, the rank of the state reduces to two) for all N, leading to $C_N(t) \sim e^{-\Gamma t}$. Consequently, the multipartite quantum correlations of W states clearly outperform those of GHZ states in terms of their robustness. One might be tempted to attribute this to
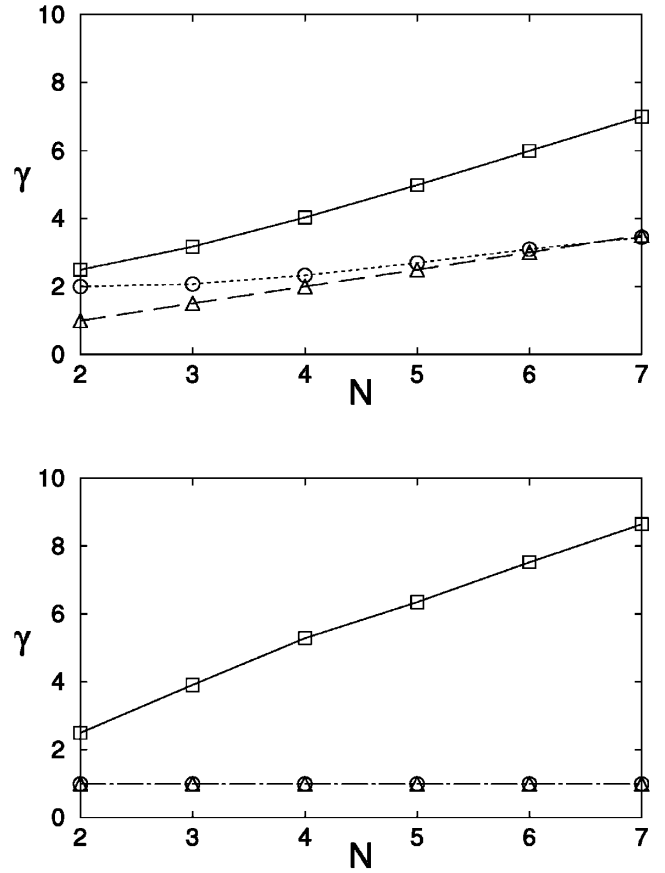

FIG. 2.   Decay rates γ (in units of the reservoir rate Γ) for GHZ (top) and W (bottom) states, as a function of the system size (i.e., the particle number) N. The different environment models are represented by circles connected by a dotted line (zero temperature), by squares connected by a solid line (infinite temperature), and by triangles connected by a long-dashed line (dephasing), respectively. While for GHZ states the decay rates increase roughly linearly with N, independently of the specific environment, the W states exhibit increasing decay with system size only for the infinite temperature environment. Remarkably, the decay rate of the W states is size independent for dephasing and zero temperature environments.

the smaller initial concurrence of W as compared to GHZ states, though the ratio

$$\frac{C_N(\Psi_{\mathrm{GHZ}})}{C_N(\Psi_W)} = \sqrt{(1 - 2^{1-N})\frac{N}{N-1}}$$

with maximum for $N = 5$ approaches unity for large N.

To summarize, we have shown that an efficient monitoring of multipartite entanglement under arbitrary (Markovian) environment coupling is possible for different classes of quasipure initial states typically dealt with in state-of-the-art experiments. Furthermore, our finding that GHZ states are significantly more fragile under environment coupling than W states might indicate a robust pathway to scalable quantum information processing.

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