Spherical $2 + p$ Spin-Glass Model: An Exactly Solvable Model for Glass to Spin-Glass Transition

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We present the full phase diagram of the spherical $2 + p$ spin-glass model with $p \geq 4$. The main outcome is the presence of a phase with both properties of full replica symmetry breaking phases of discrete models, e.g., the Sherrington-Kirkpatrick model, and those of one replica symmetry breaking. This phase has a finite complexity which leads to different dynamic and static properties. The phase diagram is rich enough to allow the study of different kinds of glass to spin glass and spin glass to spin glass phase transitions.

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In the last years many efforts have been devoted to the understanding of complex systems such as spin glasses and structural glasses, as well as optimization, biological, and financial problems. The common denominator of all these systems is a large number of stable and metastable states [1] whose complex structure determines their static or dynamic behaviors. In this framework, mean-field models, and among them spherical models, represent a valuable tool of analytical and theoretical investigation since they can be thoroughly and satisfactorily solved. Up to now only spherical models with one replica symmetry breaking (1RSB) phases were studied, mainly due to their relevance for the fragile glass transition [2–4].

To our knowledge, the possibility of full replica symmetry breaking (FRSB) phases in spherical models was first pointed out by Nieuwenhuizen [5] on the basis of the similarity between the replica free energy of some spherical models with multispin interactions and the relevant part of the free energy of the Sherrington-Kirkpatrick (SK) model [6,7]. A complete analysis, however, was not provided up to now. The problem was considered some years later [8] in connection with the possible different scenarios for the critical dynamics near the glass transition [9], therefore analyzing only the dynamical behavior in the 1RSB phase.

The Model.—The model we shall consider is the spherical $2 + p$ spin-glass model without an external field defined by the Hamiltonian

$$\mathcal{H} = \sum_{i<j} J_{ij}^{(2)} \sigma_i \sigma_j + \sum_{i_1 < \ldots < i_p} J_{i_1 \ldots i_p}^{(p)} \sigma_{i_1} \cdots \sigma_{i_p} \tag{1}$$

where $J_{i_1 i_2 .. i_p}^{(p)}$ are uncorrelated zero mean Gaussian variables of variance $J_p^2 p!/(2N^{p-1})$ and σ_i are N continuous variables obeying the spherical constraint $\sum_i \sigma_i^2 = N$. The properties of the model strongly depend on the value of p: for $p = 3$ the model reduces to the usual spherical p-spin spin-glass model in a field [3] with a low-temperature 1RSB phase, while for $p \geq 4$ the model possesses an additional FRSB low-temperature phase [5]. A partial analysis of the phase space of the model was carried out in Ref. [8] leaving out, however, a large part of the phase space and, in particular, the transition between the 1RSB and the FRSB phases.

We complete the study of the phase space focusing on the transitions lines. We have studied the model by three complementary approaches. The first employs the replica method and analyzes the disorder-averaged logarithm of the partition function following Ref. [3]. The second approach starts from the microscopic dynamics and extends the results of Ref. [8] while the latter uses the Thouless-Anderson-Palmer approach [10]. In the following we shall mainly present the replica approach, discussing differences with other approaches when necessary. A complete analysis of the properties of the model is beyond the scope of this Letter and will be presented elsewhere.

Applying the standard replica method, the free energy per spin f can be written as a function of the symmetric $n \times n$ replica overlap matrix $Q_{\alpha\beta}$ as [3]

$$-\beta f = -\beta f_0 + s(\infty) + \lim_{n\to 0} \frac{1}{n} \max_Q G[Q] \tag{2}$$

where f_0 is an irrelevant constant, $s(\infty) = (1 + \ln 2\pi)/2$ is the entropy per spin at infinite temperature $T = 1/\beta$,

$$G[Q] = \frac{1}{2} \sum_{\alpha\beta}^{1,n} g(Q_{\alpha\beta}) + \frac{1}{2} \ln\det Q, \tag{3}$$

$$g(x) = \frac{\mu_2}{2} x^2 + \frac{\mu_p}{p} x^p, \tag{4}$$

with the shorthand $\mu_p = (\beta J^{(p)})^2/2p$. The spherical constraint is ensured by the condition $Q_{\alpha\alpha} = \overline{q} = 1$.

Following Parisi [11] the overlap matrix $Q_{\alpha\beta}$ for a number R of steps in the replica symmetry breaking is divided into successive boxes of decreasing size p_r, with $p_0 = n$ and $p_{R+1} = 1$, and the elements of $Q_{\alpha\beta}$ are given by

$$Q_{\alpha\beta} \equiv Q_{\alpha\cap\beta=r} = q_r, \qquad r = 0, \cdots, R+1 \tag{5}$$

with $Q_{R+1} = \overline{q}$. The notation $\alpha \cap \beta = r$ means that α and β belong to the same box of size p_r, but to two *distinct* boxes of size $p_{r+1} < p_r$. The replica symmetric case and the FRSB case are obtained for $R = 0$ and $R \to \infty$, respectively. The matrix obtained is conveniently expressed using the function

$$x(q) = p_0 + \sum_{r=0}^{R} (p_{r+1} - p_r)\theta(q - q_r) \qquad (6)$$

which equals the fraction of a pair of replicas with overlap less or equal to q. Inserting this structure into Eqs. (2)–(4), neglecting the terms of order $O(n^2)$, and replacing the sums by integrals, one obtains, after a little of algebra,

$$-2\beta f = 2s(\infty) - 2\beta f_0 + \int_0^1 dq x(q) \frac{d}{dq} g(q) + \ln[1 - q(1)] + \int_0^{q(1)} \frac{dq}{\int_q^1 dq' x(q')} \qquad (7)$$

where $q(1) = q_R$ and $q(x)$ is the inverse of $x(q)$. Maximization of f with respect to $q(x)$ leads to the self-consistent equation(s) for the order parameter function $q(x)$. Depending on the values of $J^{(p)}$ and T, the function $q(x)$ displays different forms which characterize the different phases of the model. Figure 1 shows the phase diagram in the space of the "natural" parameters $\mu_p - \mu_2$ for $p = 4$. The results, however, are qualitatively valid for any $p \geq 4$. The stability analysis reveals four different phases, which will be discussed in the forthcoming part.

The paramagnetic phase (PM).—This phase exists for not too large values of J's and/or high temperature and is characterized by a null order parameter function. The phase becomes unstable above the line $\mu_2 = 1$ (de Almeida–Thouless line [12]) where the *replicon* $\Lambda = 1 - \mu_2$ becomes negative. In this region, for $p \geq 4$ and μ_p not too large, a FRSB phase appears. Below $\mu_2 = 1$ the PM phase remains stable for all values of μ_p, similar to what happens in the spherical p-spin model without a field [3]. However, as μ_p increases, a more thermodynamically favorable 1RSB phase with a nonvanishing order parameter takes over.

The one step replica symmetry breaking phase.— This phase is characterized by a steplike order parameter function $q(x) = q_1\theta(x - m)$ [13] and is stable as long as the replicon eigenvalue is positive:

$$\Lambda = \frac{1}{(1 - q_1 + mq_1)^2} - \frac{d^2}{dq^2} g(q)|_{q=0} > 0. \qquad (8)$$

Maximization of f with respect to q_1 and m leads to the static 1RSB equations whose solution can be conveniently expressed defining $q_1 = (1 - y)/(1 - y + my)$ and using the auxiliary function

$$z(y) = -2y \frac{1 - y + \ln y}{1 - y} \qquad (9)$$

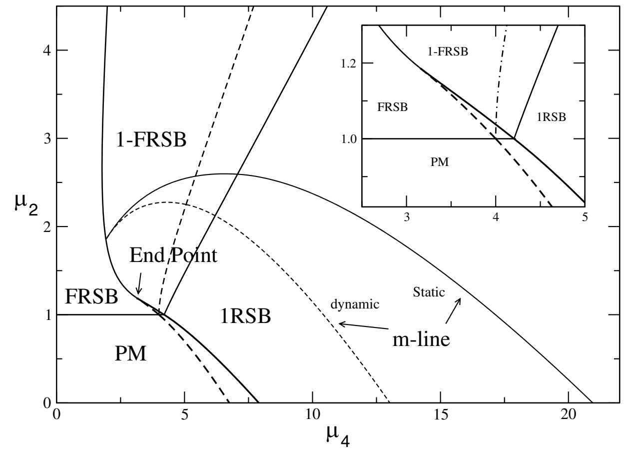

FIG. 1. Phase diagram μ_2-μ_4. Counterclockwise: the paramagnetic (PM), one step RSB (1RSB), one-full RSB (1-FRSB), and FRSB phases are plotted, separated by the *static* phase transition lines (solid curves). The PM/1RSB and the 1RSB/1-FRSB transitions also occur in the *dynamics* and the relative lines are drawn as dashed curves. The dynamic and the static PM/1RSB lines are the "m-lines" at constant $m = 1$, computed in the 1RSB ansatz [from Eq. (10)] imposing, respectively, Eq. (11) and Eq. (8). Their continuation as FRSB/1-FRSB transition lines are the dynamic (dashed) and static (solid) $m =$ 1-lines, computed in the 1-FRSB ansatz [from Eq. (13)]. They merge at the end point (see inset). For a comparison, we also plot the dynamic and static $m = 0.5$-lines. They merge on the FRSB/1-FRSB phase transition line above the end point. As m decreases from one, the whole *continuous* FRSB/1-FRSB line is covered. Inset: the discontinuous transitions FRSB/1-FRSB ($\mu_2 > 1$) and PM/1RSB ($\mu_2 < 1$).

introduced in Ref. [3] for the solution of the spherical p-spin spin-glass model. The minimum value of y is yielded by $z(y_{\min}) = 1/2$, for which $\mu_2 = 0$, while the maximum by $z(y_{\max}) = 1/2(1 + y_{\max})$, where the replicon (8) vanishes. For $p = 4$ the solution reads

$$\mu_4 = 2[1 - z(y)] \frac{(1 - y + my)^4}{m^2 y(1 - y)^2}, \qquad \mu_2 = [2z(y) - 1] \frac{(1 - y + my)^2}{m^2 y}. \qquad (10)$$

By fixing the value of $m \in [0, 1]$ and varying y from $y_{\min}$ to $y_{\max}$ one obtains the so-called m-lines. The transition between the PM and the 1RSB phases corresponds to the $m = 1$ line. Along this line q_1 jumps discontinuously from zero (PM) to a finite value (1RSB), even though the thermodynamic quantities remain continuous. Inserting $y_{\max}$ into Eq. (10) and varying m from one to zero, one obtains the critical line between the 1RSB and a different phase that we will discuss below (the 1-FRSB phase).

The static approach requires f to be maximal with respect to variations of m. The dynamics, on the other hand, leads to the different marginal condition

$$\frac{1}{(1-q_1)^2} - \frac{d^2}{dq_1^2} g(q1) = 0 \tag{11}$$

yielded by maximizing the derivative of f with respect to m (i.e., the complexity) [14]. As a consequence, the transition lines for dynamics and statics do not coincide. The complete phase diagram is plotted in Fig. 1.

The one-full replica symmetry breaking phase (1-FRSB).—The analysis of the instability of the 1RSB solution reveals that in order to stabilize the phase above the zero replicon line (8), a nonzero q_0 would be needed, but, in the absence of external fields, the order parameter function must vanish as $x \to 0$, and, hence, a 1RSB solution is not feasible. The different location of the static and dynamic instability lines, however, suggests that some sort of 1RSB-like form must survive in the solution. The way out is to look for a solution that, below q_0, has a structure which vanishes as $x \to 0$. The most general form is an order parameter

$$q(x) = \begin{cases} q_1 & \text{for x > m} \\ q_0(x) & \text{for x < m} \end{cases} \tag{12}$$

with $q(0) = 0$ and $\lim_{x \to m^-} q(x) = q_0 \neq \lim_{x \to m^+} q(x) = q(1) = q_1$; see Fig. 2. An unbiased confirmation that this is the correct ansatz follows from the numerical solution of the Parisi equations derived from the stationarity of f with respect to $q(x)$ performed in the variational Sommers-Dupont formalism [15]; the solution is yielded by means of a pseudospectral technique, see, e.g., Ref. [16]. The 1-FRSB equations are obtained by inserting the form (12) into the replica free energy (7) and imposing stationarity with respect to $q_0(x)$, q_1, and m. The resulting equations can be solved in term of m-lines similarly to what was done in the 1RSB case. For the $p = 4$ case the solution for the "discontinuous" part of $q(x)$ reads

$$\begin{aligned} \mu_4 &= \frac{[1 - y + my(1-t)]^4}{m^2 y (1-y)^2 (1-t)^3 (1+2t)}, \\ \mu_2 &= \frac{[1 - y + my(1-t)]^2}{m^2 y (1-t)^3 (1+2t)} [y(1+t+t^2) - 3t^2], \end{aligned} \tag{13}$$

$$t \equiv \frac{q_0}{q_1} = \frac{1 + y - 2z(y)}{4z(y) - 3 - y}, \tag{14}$$

and $q_1 = (1-y)/[1 - y + my(1-t)]$. The "continuous" part of $q(x)$, instead, satisfies the equation

$$q = \int_0^q dq' [\mu_2 + 3\mu_4 q'^2] \chi(q')^2, \qquad 0 \le q \le q_0 = tq_1, \tag{15}$$

$$\chi(q) = 1 - q_1 + m(q_1 - q_0) + \int_q^{q_0} dq' x(q'). \tag{16}$$

For any fixed value of $m \in [0, 1]$ these equations can be solved varying y from $y_{\min}$ (such that $x(y_{\min}) = 0$, transition line to the 1RSB phase) up to $y = 1$ ($t(y = 1) = 1$), where the difference between q_1 and q_0 vanishes. These lines are the continuation into the 1-FRSB phase of the 1RSB m-lines. In particular, the $m = 1$ line represents the transition between the 1-FRSB and the FRSB phase. For many aspects this transition is similar to the transition between the PM and the 1RSB phases: indeed $q_1 - q_0$ jumps discontinuously from a null value (FRSB) to a finite value (1-FRSB) and the discontinuity appears at $m = 1$, so that the thermodynamic quantities are continuous across the transition. The critical $m = 1$ line ends at the point where $q_1 = q_0$ (see inset of Fig. 1), which for $p = 4$ is: $q_1 = q_0 = 1/4$, $\mu_4 = (4/3)^4$, and $\mu_2 = 32/27$. Above the end point, the FRSB/1-FRSB transition occurs without an order parameter discontinuity. The value of t increases along the m-lines as one moves away from the transition line with the 1RSB phase, and the lines terminates when $t = 1$ ($q_0 = q_1$). The set of all end points for $m \in [0, 1]$ defines the continuous critical line between the two spin-glass phases:

$$\mu_4 = \frac{1}{m^2}\left(\frac{1+3m}{3}\right)^4, \qquad \mu_2 = \frac{2}{3}\left(\frac{1+3m}{3m}\right)^2. \tag{17}$$

On this line $q_1 = q_0 = 1/(1+3m)$ and $x_c \equiv x(q_0) = m$. In passing from the 1-FRSB to the FRSB phase the solution changes from stable to marginally stable.

The presence of a discontinuity in the order parameter function leads to a finite complexity and different static and dynamic solutions: the first associated with states of smallest (zero) complexity, the latter with states of largest complexity. As a consequence, the m-lines in the two cases are different, as shown in Fig. 1. The inset of the figure shows the different transition lines between the FRSB and the 1-FRSB phases. The discontinuity, and hence the complexity, vanishes at the "end point" on the continuous transition line and the two solutions coincide on this line and in the whole FRSB phase.

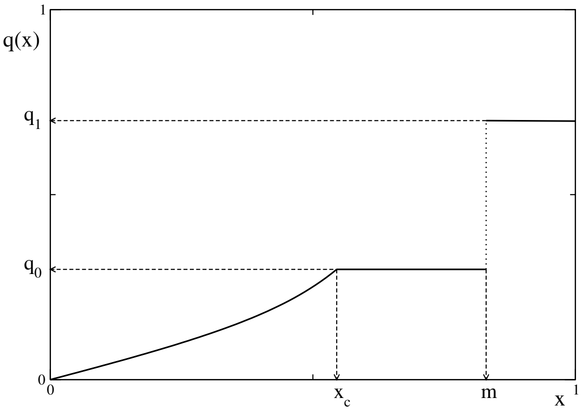

FIG. 2. Schematic form of the order parameter function $q(x)$ in the 1-FRSB phase.

The full replica symmetry breaking phase.—In this phase the order parameter function is continuous and given by Eq. (15) with $q_1 = q_0$. By expressing (15) in terms of $q(x)$, instead of $x(q)$, and taking successive derivatives with respect to x, a power expansion of $q(x)$ of about $x = 0$ can be computed [15,17]. For the $2 + 4$ model it turns out that $q(x)$ only contains odd powers of x, the first of which are

$$q(x) = \frac{\mu_2^{3/2}}{3\mu_4} x + \frac{\mu_2^{7/2}}{6\mu_4^4} x^3 + \frac{13\mu_2^{11/2}}{72\mu_4^3} x^7 + \cdots. \qquad (18)$$

One can then show that, as the PM-FRSB transition line is approached from above ($\tau = \mu_2 - 1 \to 0^+$), both $q_0 = q(x_c)$ and x_c linearly vanish with τ as

$$q_0 = \frac{\tau}{2} + O(\tau^2), \qquad x_c = \frac{3\mu_4}{2}\tau + O(\tau^2), \qquad \tau \to 0^+, \qquad (19)$$

so that the phase transition occurs continuously.

Conclusions.—We have provided the full phase diagram of the spherical $2 + p$ spin-glass model with $p \geq 4$. Despite its simplicity the model has a rather rich diagram. Not only does it present a 1RSB phase similar to the one of the spherical p-spin spin-glass model and a FRSB phase similar to that of the SK model, but it also displays a phase with an order parameter made of a continuous part for $x < m \leq 1$, and a discontinuous jump at $x = m$. To emphasize its mixed nature we call this phase, separating the FRSB phase from the 1RSB phase, the 1-FRSB phase. In many aspects it is similar to the 1RSB phase: (i) it can be proved stable [18] and (ii) it displays a finite complexity counting metastable states that are strict minima of the free energy landscape.

The FRSB/1-FRSB transition can be either continuous (for large enough μ_2) or discontinuous. In the latter case, the presence of finite complexity in the 1-FRSB phase makes the static and dynamic transition different. The two transition lines join at the end point where the discontinuity at $m = 1$ in the order parameter function vanishes. From this point on, the FRSB/1-FRSB transition only occurs continuously. Along the continuous transition line the complexity vanishes and the static and dynamic approaches lead to the same results, in agreement with the conjecture made in Ref. [21] that the complexity of the minima of the free energy landscape in the FRSB phase is zero [22].

In conclusion, we believe that this is a rather promising model since not only can it be fully solved, but it possesses different phases which can be fully analyzed. Moreover, it displays an interesting transition between two different glassy phases, similar to what is found in some colloidal suspensions [24]. Eventually, the analysis of its complexity functional can be relevant even for the comprehension of the structure(s) underlying the hard computational region in combinatorial optimization problems (see, e.g., Ref. [25]).

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