Stabilizing Near-Nonhyperbolic Chaotic Systems with Applications

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(Received 18 April 2004; published 15 November 2004)

Based on the invariance principle of differential equations a simple, systematic, and rigorous feedback scheme with the variable feedback strength is proposed to stabilize nonlinearly finite-dimensional chaotic systems without any prior analytical knowledge of the systems. Especially the method may be used to control near-nonhyperbolic chaotic systems, which, although arising naturally from models in astrophysics to those for neurobiology, all Ott-Grebogi-York type methods will fail to stabilize. The technique is successfully used for the famous Hindmarsh-Rose neuron model, the FitzHugh-Rinzel neuron model, and the Rössler hyperchaos system, respectively.

DOI: 10.1103/PhysRevLett.93.214101 PACS numbers: 05.45.Gg, 87.10.+e, 87.19.La

Since Ott, Grebogi, and Yorke (OGY) [1] proposed an effective method to control chaos, all kinds of variations based on this method have been given [2], and lots of successful experiments have been reported. For the simplicity, here we call the OGY method and its variations as the OGY-type methods. Recall the idea of the OGY-type methods, the following three steps are necessary to improve its performance: (i) To specify and locate an unstable periodic orbit embedded in the chaotic attractor, say a fixed point x_f; (ii) To approximate linearly the system in a small neighborhood of x_f by reconstructing statistically the corresponding linearized matrix J; (iii) To control (or stabilize) the chaotic orbits entering the neighborhood to x_f with aid of the approximated linear dynamics. The first step can be realized by the method of close returns from experimental data, and the third step is entirely within the field of linear control theory such as "the pole placement technique." Although the second step, including the calculation of eigenvalues and eigenvectors of the corresponding linearized matrix J, has been solved by the least-squares fit, this problem is related to how chaos affects the linear estimation of the dynamics in the small neighborhood of x_f. Especially, when the system is nonhyperbolic and any prior analytical knowledge of dynamics is not available, such linearization will be problematic due to the nonlinearity. It is well known that many of the chaotic phenomena seen in systems occurring in practice are nonhyperbolic. On the other hand, there are numerous successful reports of OGY control in numerical experiments. This matter is slightly puzzling. In [3], the author investigated carefully this problem, and found that there are two possible reasons resulting in such contradiction. One reason is that the least-squares fit used in the process of reconstructing the attractor from time series is ill defined due to the nonhyperbolicity of systems. The other is that there are large relative errors in the process of solving numerically eigenvalues of a matrix as one of its eigenvalues, $\lambda \approx 0$, which is a well-known fact in the matrix computations [4]. Therefore in those successful numerical experiments the nonhyperbolicity of system may be destroyed before obtaining the information for attempting the OGY control by experimental time series. Although the nonhyperbolicity of the chaotic attractor does not automatically mean the nonhyperbolicity of the unstable periodic orbits embedded in it, the near nonhyperbolicity must exist and affect the performance of the OGY control when the system itself is near nonhyperbolic; see the models discussed below. More interestingly, the report in [5], on the failure of chaos control in a parametrically excited pendulum whose excitation mechanism is not perfect, throws highly the light to this viewpoint (to the best of my knowledge, this is the first report on failure of the OGY control in the concrete physical experiments).

This Letter is motivated by the limitation of the OGY-type controllers as what referred above. Especially we address the control problem on the near-nonhyperbolic chaotic systems in the form of

$$\dot{u} = g_u(u, v), \qquad \dot{v} = r g_v(u, v), \tag{1}$$

where $u \in \mathbf{R}^{n_1}$, $v \in \mathbf{R}^{n_2}$, and $0 < r \ll 1$. The systems have simultaneously n_1 dramatic components u and n_2 slow variables v, which arise naturally from many scientific disciplines, and range from models in astrophysics to those for biological cells [6]. In particular, such systems and their discrete versions are widely used to model bursting, spiking, and chaotic phenomena in neuroscience; see [7] and references therein. More interestingly, just as was referred in [1], due to multipurpose flexibility of higher life form, chaos may be a necessary ingredient in their regulation by the brain. In the other side, based on a fact of cognitive science the author speculated in [3] that such chaotic ingredient is probably in the form of (1), where the slow variables represent a "container" or "recorder" storing the acquired knowledge. Note that all OGY-type methods will fail to control such chaotic dynamics because system (1) with sufficiently small r is near-nonhyperbolic, i.e., the corresponding linearized matrix J at any points admits one eigenvalue $\lambda \approx 0$. For the discrete case this indicates that the linearized matrix J admits one eigenvalue $\lambda \approx 1$. However, all OGY-type controllers contain the term $(J - I)^{-1}$ (see [2], where I is the identity matrix), which is near singular in this case, so all OGY-type methods are infeasible for system (1) with $0 < r \ll 1$. This is just the reason for failure of chaos control reported in [5], meanwhile this may be also a reason why it is so difficult to physically control chaos in the brain by the OGY method [8]. In addition, this mechanism is beneficial to explain why stabilization of an inverted triple pendulum is very troublesome as out-of planar motions become very substantial, which was firstly reported in [9].

In this Letter, based on the invariance principle of differential equations [10], a simple and rigorous feedback scheme with the variable feedback strength is proposed to stabilize nonlinearly finite-dimensional chaotic and hyperchaotic systems without any prior analytical knowledge of the systems. Especially, this simple technique can be easily applied to stabilize near-non-hyperbolic chaotic systems in the form of (1). This Letter is mainly focused on the continuous systems, but the proposed method can be generalized to the discrete version by the invariance principle of difference equations.

Let a chaotic system be given as

$$\dot{x} = f(x), \tag{2}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$, $f(x) = (f_1(x), f_2(x), \cdots, f_n(x)):\mathbf{R}^n \rightarrow \mathbf{R}^n$ is a nonlinear vector function. Without loss of the generality we let $\Omega \subset \mathbf{R}^n$ be a chaotic bounded set of (1) which is globally attractive, and suppose that $x = 0$ is a fixed point embedded in Ω. For the vector function $f(x)$, we give a general assumption.

For any $x = (x_1, x_2, \cdots, x_n) \in \Omega$, *there exists a constant* $l > 0$ *satisfying*

$$| f_i(x) | \leq l \max_j | x_j |, \qquad i = 1, 2, \cdots, n. \tag{3}$$

Note this condition is very loose, for example, the condition (3) holds as long as $(\partial f_i/\partial x_j)(i, j = 1, 2, \cdots, n)$ are bounded. Therefore the class of systems in the form of (2) and (3) include almost all well-known finite-dimensional chaotic and hyperchaotic systems. To stabilize the chaotic orbits in (2) to the fixed point $x = 0$, we consider the feedback control

$$\dot{x} = f(x) + \epsilon x, \tag{4}$$

where $\epsilon x = (\epsilon_1 x_1, \epsilon_2 x_2, \cdots, \epsilon_n x_n)$. Instead of the usual linear feedback, the feedback strength $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ here will be duly adapted according to the following update law:

$$\dot{\epsilon}_i = -\gamma_i x_i^2, \qquad i = 1, 2, \cdots, n, \tag{5}$$

where $\gamma_i > 0$, $i = 1, 2, \cdots, n$, are arbitrary constants. For the system consisting of (4) and (5), we introduce the following function:

$$V = \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(\epsilon_i + L)^2, \tag{6}$$

where L is a constant bigger than nl, i.e., $L > nl$. By differentiating the function V along the trajectories of system (4) and (5), we obtain

$$\dot{V} = \sum_{i=1}^{n} x_i(f_i(x) + \epsilon_i x_i) - \sum_{i=1}^{n}(\epsilon_i + L)x_i^2 \leq (nl - L)\sum_{i=1}^{n} x_i^2 \leq 0,$$

where we have assumed $x \in \Omega$ (without loss of the generality as Ω is globally attractive), and used the condition (3). It is obvious that $\dot{V} = 0$ if and only if $x = 0$, namely, the set $E = \{(x, \epsilon) \in \mathbf{R}^{2n} : x = 0, \epsilon = \epsilon_0 \in \mathbf{R}^n\}$ is the largest invariant set contained in $\dot{V} = 0$ for systems (4) and (5). Then according to the well-known invariance principle of differential equations [10], starting with arbitrary initial values of systems (4) and (5), the orbit converges asymptotically to the set E, i.e., $x \rightarrow 0$ and $\epsilon \rightarrow \epsilon_0$ as $t \rightarrow \infty$.

Obviously when the chaotic system (2) is stabilized to $x = 0$ the variable feedback strength ϵ will be automatically adapted to a suitable strength ϵ_0 depending on the initial values. This is significantly different from the usual linear feedback, and the converged strength must be of the lower order than those used in the constant gain schemes. Although theoretically the converged strength may be very big so that it may give rise to its own dynamics, the flexibility of the strength in the proposed scheme can overcome this limitation once such a case arises. For example, suppose that the feedback strength is restricted not to exceed a critical value, say k. In the present control procedure, once the variable strength ϵ exceeds k at time $t = t_0$, we may choose the values of variables at this time as initial values and repeat the same control by resetting the initial strength $\epsilon(0) = 0$. Namely, one may achieve the stabilization within the restricted feedback strength due to the global stability of the present scheme. This idea is slightly similar to that of the OGY control [1], i.e., small parameter control. But there exists a certain difference between them, for example, in the OGY control the controller waits passively for the emergence of chaotic orbits. In the other side, in the present adaptive-feedback scheme the small converged strength may be obtained by adjusting suitably the parameter γ. Moreover, we note that in the present scheme it is not necessary for some particular models to use all the variables of system as feedback signals. For example, if $| e_i | \leq | e_j |$ one may set $\epsilon_i \equiv 0$ which implies that it does not need to add the feedback control to the variable x_i at all, and this case exists in general due to the nonhyperbolicity of chaotic attractor, see the following examples. Obviously this simple, systematic and rigorous method may stabilize nonlinearly almost all finite-dimensional chaotic systems [including those in the form of (1)] without any *a priori* analytical knowledge of systems, and is

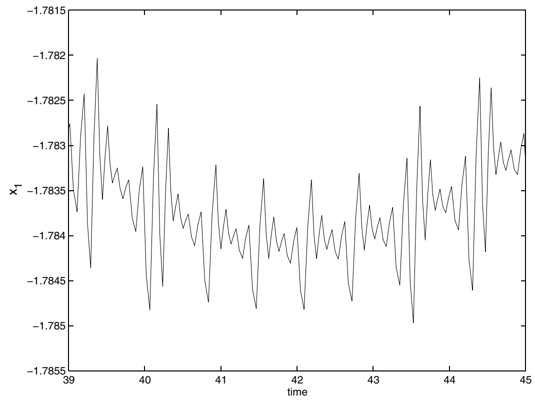

FIG. 1. Time series $x_1(t)$ generated by the chaotic Hindmarsh-Rose model (8).

robust against the effect of noise due to the global nonlinear stability.

Next we will give three illustrative examples. We take the famous Hindmarsh-Rose neuron model [11] as the first example, which is governed by the following third-order ordinary differential equation:

$$\dot{x}_1 = x_2 + 3x_1^2 - x_1^3 - x_3 + I, \quad \dot{x}_2 = 1 - 5x_1^2 - x_2, \\ \dot{x}_3 = -rx_3 + 4r(x_1 + 1.6), \tag{8}$$

with $0 < r \ll 1$. Here x_1 is the membrane potential of the neuron, x_2 is a recovery variable, and x_3 is a slow adaptation current. It has been found in [12] that the model admits a chaotic attractor with $r = 0.0012$ and the external current $I = 3.281$, see Fig. 1 for the chaotic time series of x_1. After transforming the unique fixed point $(-0.6835, -1.3359, 3.666)$ to $(0, 0, 0)$, we stabilize successfully this near-nonhyperbolic chaotic system by the

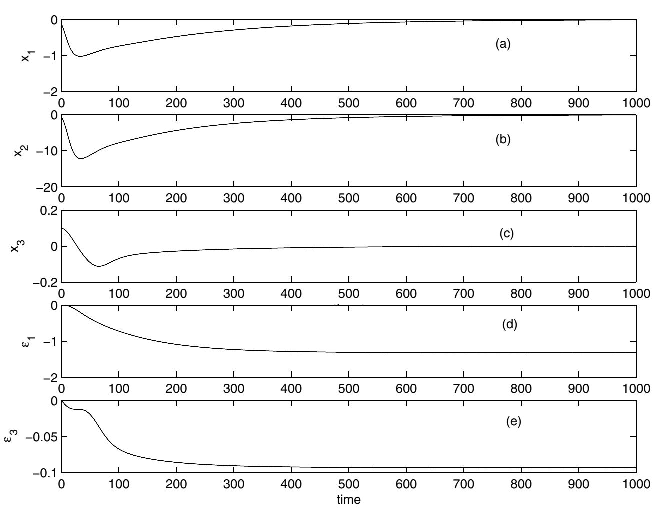

FIG. 2. The chaotic Hindmarsh-Rose model (8) is stabilized successfully by only two feedback signals, i.e., x_1 and x_3, where (a)–(c) show the temporal evolution of the variables x_i, $i = 1$, 2, 3, and (d)–(e) correspond to the variable feedback strength ϵ_1 and ϵ_2.

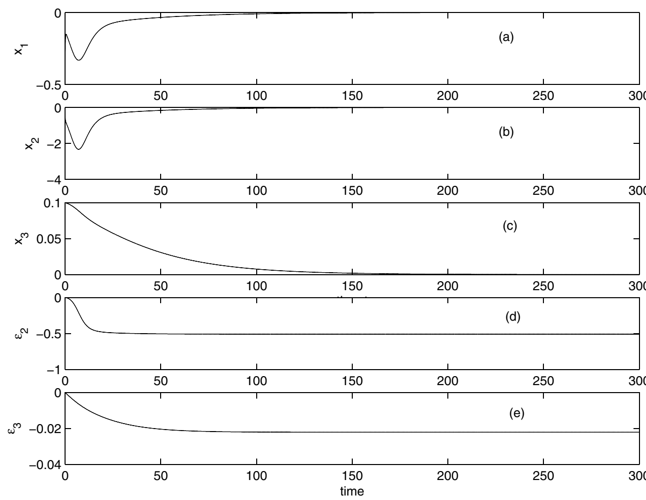

FIG. 3. The chaotic Hindmarsh-Rose model (8) may also be stabilized by another two feedback signals, i.e., x_2 and x_3.

proposed scheme, where the feedback control as (4) is not added to the variable x_2, i.e., letting $\epsilon_2 \equiv 0$. The corresponding numerical results and the evolution of ϵ are shown in Fig. 2, where the initial values are set as $(-0.5, -0.3, 0.1, 0, 0)$, and the parameters $\gamma_1 = 0.01$, $\gamma_3 = 0.1$. In addition, in Fig. 3 we show that this chaotic system may also be stabilized by another two feedback signals, i.e., x_2 and x_3, where all initial values are same those in Fig. 2 and parametrical values are set as $\gamma_2 = 0.01$, $\gamma_3 = 0.1$. However, we find numerically that such stabilization is troublesome by the other feedback signals, e.g., x_1 and x_2, which confirms difficulty of stabilizing the near-nonhyperbolic chaotic systems provided ignoring such near-nonhyperbolicity (respectively, the slow variable).

The second illustrative example is the FitzHugh-Rinzel neuron model [13]:

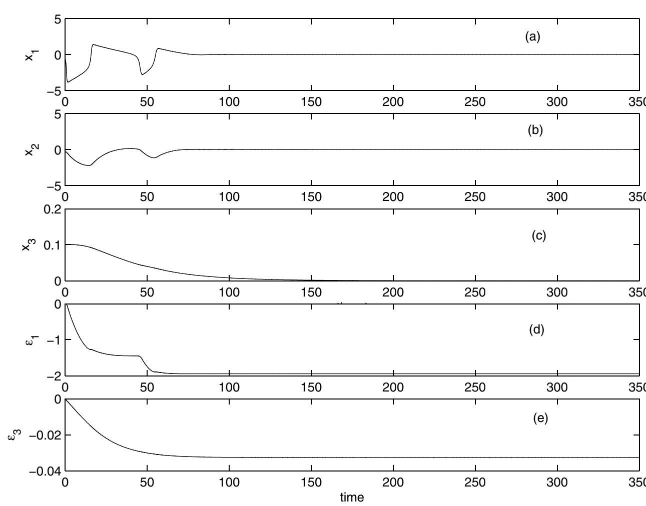

FIG. 4. The FitzHugh-Rinzel neuron model (9) is stabilized successfully by only two feedback signals, i.e., x_1 and x_3, where all parameters and initial values are same as those used in Fig. 2.

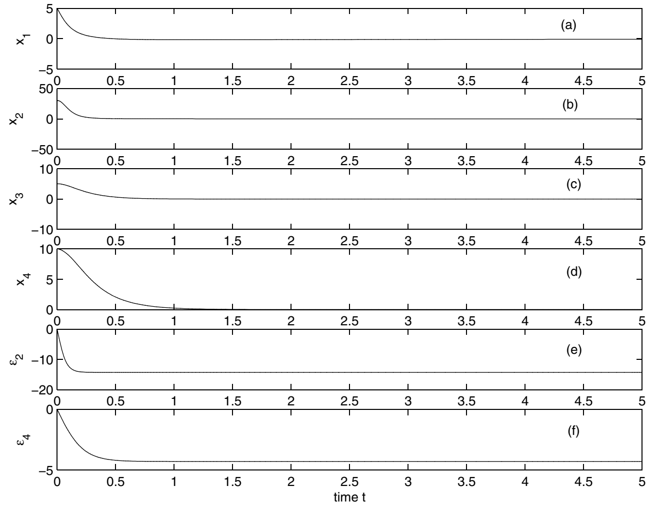

FIG. 5. The Rössler hyperchaotic system (10) is stabilized by only two feedback signals, i.e., x_2 and x_4, where (a)–(d) show the temporal evolution of the variables x_i, $i = 1, 2, 3, 4$, and (e)–(f) correspond to the temporal evolution of feedback strength ϵ_2 and ϵ_4.

$$\dot{x}_1 = x_1 - \frac{1}{3}x_1^3 - x_2 + x_3 + 0.3125, \qquad (9)$$
$$\dot{x}_2 = 0.08(0.7 + x_1 - 0.8x_2), \quad \dot{x}_3 = -r(0.775 + x_3 + x_1),$$

where $r = 0.0001$. After transforming the unique fixed point $(-0.885, -0.231, 0.11)$ to $(0, 0, 0)$, we stabilize successfully this near-nonhyperbolic chaotic model by the proposed adaptive-feedback scheme, see Fig. 4, where all parameters and initial values are same as those used in Fig. 2 in the first example.

(*Remark.*—The reason why the initial values of x in Figs. 2–4 look unlike the given values $(-0.5, -0.3, 0.1)$ is that these variables vary very quickly in the initial interval of time while the whole time interval is too long).

To show the generality of the present method, our final example is the famous Rössler hyperchaos system:

$$\dot{x}_1 = -x_2 - x_3, \qquad \dot{x}_2 = x_1 + 0.25x_2 + x_4,$$
$$\dot{x}_3 = 3 + x_1x_3, \qquad \dot{x}_4 = -0.5x_3 + 0.05x_4. \qquad (10)$$

Similarly after transforming the unique fixed point $(-5.4083, -0.5547, 0.5547, 5.547)$ to $(0, 0, 0, 0)$, the hyperbolic chaotic system is stabilized by the present method, where the feedback control as (4) is added to only the variable x_2, i.e., letting $\epsilon_1 = \epsilon_3 \equiv 0$. The corresponding numerical results and the evolution of ϵ are shown in Fig. 5, where the initial values are set as $(5, 30, 5, 10, 0, 0)$, and the parameters $\gamma_2 = \gamma_4 = 0.2$.

In conclusion, we have given a simple, systematic, and rigorous method to stabilize nonlinearly finite-dimensional chaotic systems, which does not need any trial and error, physical intuition, and empirical determination of proportionality factor in comparison with the previous methods. Especially the method may be used to stabilize the near-nonhyperbolic chaotic systems, which all OGY-type methods will fail to stabilize. The proposed method may be used to stabilize chaos into unstable periodic orbits especially when the idea is extended to discrete version by the invariance principle of difference equations, which maybe helps us with comprehending multipurpose flexibility of the brain. In addition, this idea of control has been applied successfully to chaotic synchronization by the author [14], so we believe that the idea may be used to explore the interesting dynamical properties found in neurobiological systems, i.e., the onset of regular bursts in a group of irregularly bursting neurons with different individual properties [15].

This work is supported by The National Natural Science Foundation of China (10201020; 10432010).

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