Berry Curvature on the Fermi Surface: Anomalous Hall Effect as a Topological Fermi-Liquid Property

F. D. M. Haldane

Department of Physics, Princeton University, Princeton New Jersey 08544-0708, USA

(Received 28 June 2004; revised manuscript received 20 October 2004; published 11 November 2004)

The intrinsic anomalous Hall effect in metallic ferromagnets is shown to be controlled by Berry phases accumulated by adiabatic motion of quasiparticles on the Fermi surface, and is purely a Fermi-liquid property, not a bulk Fermi sea property like Landau diamagnetism, as has been previously supposed. Berry phases are a new topological ingredient that must be added to Landau Fermi-liquid theory in the presence of broken inversion or time-reversal symmetry.

DOI: 10.1103/PhysRevLett.93.206602 PACS numbers: 72.15.–v, 73.43.–f

Renewed interest in the anomalous Hall effect (AHE) in metallic ferromagnets has lead to a reinterpretation of the classic Karplus-Luttinger formula [1] for the "anomalous velocity" in terms of the Berry curvature of occupied electronic Bloch states [2,3]. This gives an intrinsic contribution to the Hall conductivity in the low-temperature clean limit of metallic ferromagnets when the quasiparticle lifetimes become long, and now appears to be the dominant contribution to the AHE [4,5].

The expression [1–3] for the intrinsic Hall conductivity of metals with broken time-reversal symmetry has the all the appearance of a "bulk" band structure property that depends on all the filled electronic states, not just the ones at the Fermi level. However, this seems at odds with the spirit of Landau's Fermi-liquid theory, which holds that charge transport in metals involves only quasiparticles with energies within $k_B T$ of the Fermi level.

In this Letter, I show that, despite appearances, the *nonquantized part* of the intrinsic Hall conductivity is completely expressible in terms of Berry phases [6] of quasiparticles moving on the Fermi surface, and thus fully consistent with Fermi-liquid theory. This exposes a new topological ingredient that must be added to Fermi-liquid theory unless both inversion and time-reversal symmetry are present: quasiparticle Berry phases.

The "anomalous velocity" is an extra term [7,8] in the semiclassical equations of motion of a Bloch electron in weak electric and magnetic fields; ignoring Zeeman couplings, these are

$$\hbar \frac{dk_a}{dt} = eE_a(\mathbf{x}) + eF_{ab}(\mathbf{x})\frac{dx^b}{dt}, \tag{1}$$

$$\frac{dx^a}{dt} = \frac{1}{\hbar}\nabla_k^a \varepsilon_n(\mathbf{k}) + \mathcal{F}_n^{ab}(\mathbf{k})\frac{dk_b}{dt}, \tag{2}$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a = \epsilon_{abc} B^c$ is the magnetic flux density written as an antisymmetric tensor, $\nabla_a \equiv \partial/\partial x^a$, $\varepsilon_n(\mathbf{k})$ is the energy of a Bloch electron in band n, $\nabla_k^a \equiv \partial/\partial k_a$ and $\mathcal{F}_n^{ab}$ is the antisymmetric "Berry curvature" tensor in $\mathbf{k}$ space, described below. The anomalous velocity in (2) is the $\mathbf{k}$-space dual of the Lorentz force.

If one writes the electron occupations $\langle n_{kn}\rangle$ as $n_n^0(\mathbf{k}, \mu) + \delta n_{kn}$, where $n_n^0(\mathbf{k}, \mu)$ is the ground-state occupation function at chemical potential μ, the linear current response to a uniform electric field $\mathbf{E}$ (with $\mathbf{B} = 0$) is

$$J_e^a = \frac{e}{\hbar}\frac{1}{\Omega N}\sum_{kn}\nabla_k^a \varepsilon_n(\mathbf{k})\delta n_{kn} + \sigma_0^{ab}(\mu)E_b, \tag{3}$$

where N is the number of primitive unit cells, which have volume Ω. Here $\sigma_0^{ab}(\mu)$ is an intrinsic ground-state property describing a dissipationless Hall conductivity

$$\sigma_0^{ab}(\mu) = \frac{e^2}{\hbar}\frac{1}{\Omega N}\sum_{kn}\mathcal{F}_n^{ab} n_n^0(\mathbf{k}, \mu). \tag{4}$$

This also controls the low-temperature limit of the thermal Hall conductivity κ^{ab} (Righi-Leduc effect) and the Peltier coefficient α^{ab}; their "intrinsic" parts are

$$\kappa_0^{ab}(\mu) = \frac{\pi^2}{3}\frac{k_B^2 T}{e^2}\sigma_0^{ab}(\mu), \qquad \alpha_0^{ab}(\mu) = e\frac{\partial \kappa_0^{ab}(\mu)}{\partial \mu}. \tag{5}$$

A heat current $J_Q^a = T\alpha_0^{ab}E_b$ flows with the Hall current.

If time-reversal symmetry is present, the electronic bands have the property

$$\varepsilon_n(-\boldsymbol{k}) = \varepsilon_n(\boldsymbol{k}), \qquad \mathcal{F}_n^{ab}(-\boldsymbol{k}) = -\mathcal{F}_n^{ab}(-\boldsymbol{k}), \tag{6}$$

the sum (4) cancels, and the intrinsic Hall conductivity vanishes. If inversion symmetry is also unbroken, $\mathcal{F}_n^{ab}(-\boldsymbol{k}) = \mathcal{F}_n^{ab}(\boldsymbol{k})$, and the Berry curvature vanishes.

The Berry curvature is obtained from a "vector potential" derived from the 1-particle Bloch states $|\psi_n(\boldsymbol{k})\rangle$:

$$\mathcal{A}_n^a(\boldsymbol{k}) = -i\langle\psi_n(\boldsymbol{k})|\nabla_k^a\psi_n(\boldsymbol{k})\rangle, \tag{7}$$

$$\mathcal{F}_n^{ab}(\boldsymbol{k}) = \nabla_k^a \mathcal{A}_n^b(\boldsymbol{k}) - \nabla_k^b \mathcal{A}_n^a(\boldsymbol{k}), \tag{8}$$

$$\epsilon_{abc}\nabla_k^a \mathcal{F}_n^{bc}(\boldsymbol{k}) = \sum_i q_{ni}\delta^3(\boldsymbol{k} - \boldsymbol{k}_{ni}), \qquad q_{ni} = \pm 2\pi. \tag{9}$$

The last equation is the divergence of the $\boldsymbol{k}$-space Berry curvature field $\mathcal{F}_n^{ab}(\boldsymbol{k})$: it is divergence-free except for quantized "monopole" sources with a "charge quantum" 2π, which are associated with band degeneracies. These occur at isolated $\boldsymbol{k}$ points: in complex Hermitian eigenproblems, it is sufficient to vary three parameters (here, the components of $\boldsymbol{k}$) to encounter degeneracies.

Assuming that the one-electron energy $\varepsilon_n(\boldsymbol{k})$ is nondegenerate, the wave functions are completely defined, except for an arbitrary phase factor. If both time-reversal and inversion symmetry are present, this can consistently be chosen real, but otherwise, is an arbitrary complex factor that can vary continuously with $\boldsymbol{k}$. The Berry vector potential $\mathcal{A}_n^a(\boldsymbol{k})$ depends on this "gauge choice" but the curvature $\mathcal{F}_n^{ab}(\boldsymbol{k})$ is a well-defined gauge-invariant quantity with physical significance. The Berry phase [6] for a closed path Γ is also gauge invariant:

$$\exp i\phi_n(\Gamma) = \exp i \oint_\Gamma \mathcal{A}_n^a(\boldsymbol{k}) dk_a. \tag{10}$$

The integral is the curvature flux linked through Γ: the 2π ambiguity of the Berry phase quantizes the "charge" of the monopole sources of the curvature field.

In a 3D band structure, the integral over the Brillouin zone (BZ) of the Berry curvature of a nondegenerate band is a topological invariant [9] that is a generalization of the better-known 2D Chern number [10,11]

$$\frac{1}{2\pi}\int d^3\boldsymbol{k}\,\mathcal{F}_n^{ab}(\boldsymbol{k})P_{\mathrm{BZ}}(\boldsymbol{k}) = C_n\epsilon^{abc}G_{cn}^C, \tag{11}$$

where C_n is an integer Chern number, $1/2\pi$ times the integral of the Berry curvature over a compact 2D surface, e.g., a 2D BZ (a torus), and $\mathbf{G}_n^C$ is a primitive reciprocal lattice vector; $P_{\mathrm{BZ}}(\boldsymbol{k}) = 1$ inside the BZ, 0 outside it.

The intrinsic Hall conductivity may be parametrized as $\sigma_0^{ab} = (e/\Phi_0)\epsilon^{abc}(K_c/2\pi)$, $\Phi_0 = h/e$, where $\boldsymbol{k}$ is dimensionally a wave vector: in one-electron band theory,

$$\frac{1}{2\pi}\sum_n \int d^3\boldsymbol{k}\,\mathcal{F}_n^{ab}P_{\mathrm{BZ}}(\boldsymbol{k})n_n^0(\boldsymbol{k},\mu) = \epsilon^{abc}K_c. \tag{12}$$

If band n is completely below the Fermi level, it contributes a quantized amount $C_n\boldsymbol{G}_n^C$ to $\boldsymbol{k}$. This produces an integer quantum Hall effect (QHE) with "filling factor" $\nu = C_n$ on the lattice planes indexed by $\boldsymbol{G}_n^C$.

The QHE is usually discussed in the context of strong magnetic fields where the electronic states are split up into Landau levels, so it might be wondered how "simple" Bloch electrons could exhibit a QHE without Landau levels. In fact, only broken time-reversal symmetry is required: the possibility of a $\mathbf{B} = 0$ "zero-field QHE" was first demonstrated in Ref. [12], using a model that, in retrospect, exhibits both a nonmetallic QHE phase and a metallic AHE phase.

It will be useful to understand the process by which the Chern invariants of a 3D band structure can change. While a band remains nondegenerate, its Chern invariant is "quantized" to be a reciprocal lattice vector. As a control parameter is varied, two bands may come into contact at some point in the BZ, and this initial degeneracy point then subsequently splits into two "Dirac point" singularities (near which the energy dispersion is linear). The bands are now tightly coupled by a "Berry flux loop" where Berry curvature flux 2π passes from one band to the other through one Dirac point, then returns through the other one. There is a striking analogy to the idea of "wormholes" connecting different universes, where here the two "universes" are Bloch bands, and "space" is $\boldsymbol{k}$ space. Each band has one positive and one negative monopole source of Berry curvature; at each Dirac point the two bands have opposite-sign sources. Eventually, after relative displacement by a reciprocal lattice vector $\boldsymbol{G}$, the monopoles may recombine, allowing the bands to split apart. This process conserves the sum of their invariants, but individually they change by $\pm\boldsymbol{G}$.

It is useful to first examine (4) in the simpler 2D case. Then $\sigma_0^{xy} = \nu e/\Phi_0$, where $\nu = \sum_n \nu_n$, and

$$\nu_n(\mu) = \frac{1}{2\pi}\int d^2\boldsymbol{k}\,\mathcal{F}_n^{xy}(\boldsymbol{k})P_{\mathrm{BZ}}(\boldsymbol{k})n_n(\boldsymbol{k},\mu). \tag{13}$$

For simplicity, assume that the occupied region does not touch the Brillouin zone boundary (BZB), and drop $P_{\mathrm{BZ}}(\boldsymbol{k})$. Using the Berry vector potential representation and integrating by parts gives

$$\nu_n = \frac{1}{2\pi}\int d^2\boldsymbol{k}\big(\mathcal{A}_n^x\nabla_k^y n_n^0(\boldsymbol{k}) - A_n^y\nabla_k^x n_n^0(\boldsymbol{k})\big), \tag{14}$$

which is clearly a Fermi-surface integral if the band is partially filled, since $n_n^0(\boldsymbol{k})$ has a step discontinuity at the Fermi surface, and is constant everywhere else. If the Fermi surface is a simple closed loop, this can be recognized as the integral giving the Berry phase ϕ_F for an adiabatic path around the Fermi surface (the $\boldsymbol{k}$-space version of the Bohm-Aharonov effect)

$$\nu_n = \frac{1}{2\pi}\oint \mathcal{A}_n^a(\boldsymbol{k}_F)d\boldsymbol{k}_{Fa} = \frac{\phi_F}{2\pi}. \tag{15}$$

Since the Berry phase is ambiguous by a multiple of 2π, only the nonquantized part of the intrinsic Hall conductivity is determined at the Fermi surface. If the system has evolved adiabatically along some path in a parameter space from one with time-reversal symmetry, the QHE can be determined from the history of ϕ_F during that process.

Armed with the insight that the AHE is a Fermi-surface property, I now examine the 3D problem. In general, there may be multiple sheets S_α of the Fermi surface, with both simple and multiply connected topology, with pieces "glued together" at degeneracy points where a line of high-symmetry intersects the surface, or along lines where a plane of high-symmetry intersects it; if time-reversal symmetry is unbroken, there may also be

Kramers degeneracy [13]. Complexes of intrinsically connected sheets $\boldsymbol{k}_F^{(\alpha)}(\boldsymbol{s})$, where $\boldsymbol{s} = \{s^1, s^2\}$ is a surface parametrization, will be referenced by a single label α, specified implicitly if $\mathbf{s} \in S_\alpha$. The outward normal unit vector $\hat{\boldsymbol{n}}(\boldsymbol{s})$ is also the direction of the Fermi velocity.

The key Fermi-surface property is the Luttinger sum rule relating particle number to Fermi-surface volume. The Fermi-surface geometry fixes the particle density modulo integer multiples of the "density quantum" $\rho_0 = 1/\Omega$ associated with filled bands. The change in particle density $\delta\rho_\alpha(\boldsymbol{r})$ associated with a local fluctuation $\delta\boldsymbol{k}_F(\boldsymbol{r}, \boldsymbol{s}) = \boldsymbol{\nabla}\varphi(\boldsymbol{r}, \boldsymbol{s})$ [where $\varphi(\boldsymbol{r}, \boldsymbol{s})$ is the quasiparticle phase] is just proportional to the $\boldsymbol{k}$-space volume swept out by the changing Fermi surface:

$$\partial_\mu \boldsymbol{k}_F(\boldsymbol{s}) \times \partial_\nu \boldsymbol{k}_F(\boldsymbol{s}) = \ell_{\mu\nu}(\boldsymbol{s})\hat{\boldsymbol{n}}(\boldsymbol{s}), \tag{16}$$

$$\delta\rho_\alpha(\boldsymbol{r}) = \int_{S_\alpha} \frac{ds^\mu \wedge ds^\nu}{(2\pi)^3} \ell_{\mu\nu}(\boldsymbol{s})\hat{\boldsymbol{n}}(\boldsymbol{s}) \cdot \boldsymbol{\nabla}\varphi(\boldsymbol{r}, \boldsymbol{s}), \tag{17}$$

$$\int_{S_\alpha} \frac{ds^\mu \wedge ds^\nu}{(2\pi)^3} \ell_{\mu\nu}(\boldsymbol{s})\hat{\boldsymbol{n}}(\boldsymbol{s}) = \rho_0 \sigma_\alpha \boldsymbol{R}_\chi. \tag{18}$$

Here $\partial_\mu \equiv \partial/\partial s^\mu$; a nonzero integer σ_α signals a "chiral anomaly": the system then has a quasi-1D character where the primitive real-space lattice vector $\boldsymbol{R}_\chi$ defines a special direction of lattice lines along which quasi-1D electrons predominantly move (such a band structure may also have nonchiral Fermi-surface sheets.)

The change in density $\delta\rho_\alpha$ if the Fermi-surface sheet α is rigidly displaced by a constant shift $\delta\boldsymbol{k}_F$ is $\sigma_\alpha\rho_0\boldsymbol{R}_\chi \cdot \delta\boldsymbol{k}_F$. Note that the absolute value of $\boldsymbol{k}_F(\boldsymbol{s})$ is not invariant under position-space gauge transformations $\boldsymbol{k}_F \to \boldsymbol{k}_F - (e/\hbar)\boldsymbol{A}(\boldsymbol{r})$ and only the relative displacements (modulo reciprocal lattice vectors) between Fermi vectors are physically meaningful. Gauge invariance requires that the total Fermi-surface chiral anomaly must vanish. Typical Fermi-surface sheets are nonchiral but the possibility of chiral sheets needs to be kept in mind; the integer σ_α is a measure of how many distinct chiral sheets are glued together.

It is straightforward to repeat the integration by parts to expose the nonquantized part of the 3D intrinsic Hall conductivity as a Fermi-surface property. The Berry vector potential and associated gauge-invariant Berry curvature for paths restricted to lie in a surface $\boldsymbol{k}(\boldsymbol{s})$ are

$$\mathcal{A}_\mu(\boldsymbol{s}) = \mathcal{A}^a(\boldsymbol{k}(\boldsymbol{s}))\partial_\mu k_a(\boldsymbol{s}), \quad \mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu. \tag{19}$$

The surface integral that results from integration by parts of the band-n contribution is over a surface S_n that is divided into a set of one or more outward-oriented compact surfaces enclosing occupied regions within the BZ:

$$\boldsymbol{K}_n = \frac{1}{2\pi} \int_{S_n} ds^\mu \wedge ds^\nu \mathcal{F}_{\mu\nu}(\boldsymbol{s})\boldsymbol{k}(\boldsymbol{s}). \tag{20}$$

Some parts of S_n are on the Fermi surface, but there may also be contributions from states below the Fermi level which are on the BZ boundary (BZB). The Fermi surface of band n will be divided into one or more disjoint oriented surfaces $S_{n\alpha}$, and the intersections of these with the BZB define a set of conjugate pairs of closed directed paths $C_{n\alpha i\pm}$ which have Berry phases $\pm\phi_{n\alpha i}$; the "+" path is displaced relative to its partner on the opposite side of the BZB by a primitive reciprocal lattice vector $\boldsymbol{G}_{n\alpha i}$. The contribution to the integral (20) from conjugate BZB intersections is $\phi_{n\alpha i}\boldsymbol{G}_{n\alpha i}/2\pi$, which can be apportioned equally to the "+" and "−" paths.

Eliminating the integrals over the BZB allows band indices n to be dropped, and $\sum_n \boldsymbol{K}_n$ can be written (modulo $\boldsymbol{G}$) as a sum $\sum_\alpha \boldsymbol{K}_\alpha$ of Fermi-surface integrals

$$\boldsymbol{K}_\alpha = \frac{1}{2\pi} \int_{S_\alpha} d^2\mathcal{F}\,\boldsymbol{k}_F + \frac{1}{4\pi} \sum_i \boldsymbol{G}_{\alpha i} \int_{\partial S_\alpha^i} d\mathcal{A}, \tag{21}$$

where $d^2\mathcal{F} \equiv \mathcal{F}_{\mu\nu}(\boldsymbol{s})ds^\mu \wedge ds^\nu$ is the Berry curvature 2-form and $d\mathcal{A}$ is the connection 1-form $\mathcal{A}_\mu(\boldsymbol{s})ds^\mu$; ∂S_α^i are the 1-manifolds where S_α intersects the BZB, across which $\boldsymbol{k}_F(\boldsymbol{s})$ jumps by $\boldsymbol{G}_{\alpha i}$. These boundary terms are Berry-gauge dependent, but if S_α is nonchiral, the BZ can be chosen so all ∂S_α^i are closed paths, and the gauge ambiguity is an (unknown) quantized QHE contribution. [If S_α is chiral, (21) is valid in a Berry gauge where $\mathcal{A}_\mu(\boldsymbol{s})$ is periodic, and the boundary terms cancel.]

The integral of $1/2\pi$ times the Berry curvature over Fermi-surface sheet α defines an integer Chern number C_α. If a gauge transformation shifts $\boldsymbol{k}_F(\boldsymbol{s})$ by a constant $\delta\boldsymbol{k}_F$, $\boldsymbol{k}_\alpha$ changes by $C_\alpha\delta\boldsymbol{k}_F$. Gauge invariance requires that the total sum of Fermi-surface Chern numbers must vanish.

If a nonchiral compact piece of Fermi surface has nonzero Chern number, it must enclose a source of Berry curvature, and has a hidden "wormhole" connection through which "spectral flow" can occur. Consider two bands linked by a "Berry flux loop" through a pair of Dirac points, one below the Fermi level, the other above it (both time-reversal and inversion symmetries must be broken). A particlelike Fermi surface with Chern number $C = \pm 1$ surrounds the lower Dirac point, and a holelike one with opposite-sign Chern number surrounds the other. A "spectral flow" process driven by spatial inhomogeneities can "pump" states into a band through one Dirac point and out through the other, conserving the total number of states per band. States that flow carry their occupations with them: the volumes of the holelike and particlelike Fermi surfaces shrink or expand by the same amount, conserving total charge.

The additional Berry phase terms in (21) are there for a very concrete reason: the choice of the Brillouin zone defined by $P_{\mathrm{BZ}}(\boldsymbol{k})$ is yet another kind of arbitrary gauge choice: since $\boldsymbol{k}_F(\boldsymbol{s})$ is defined to be in the BZ, however it is chosen, there must be BZB lines on a multiply connected surface across which $\boldsymbol{k}_F(\boldsymbol{s})$ jumps discontinuously back

into the BZ. The Berry phase counterterms merely guarantee that the value of $\boldsymbol{K}_\alpha$ is unchanged by any continuous deformation of the standard BZ into any other primitive cell.

The Hall conductivity also controls the charge density $\rho_e = e\delta\rho$ induced when a uniform magnetic flux density is applied, keeping the Fermi energy μ fixed:

$$\lim_{\boldsymbol{B}\to 0} \frac{\partial \rho_e}{\partial F_{ab}} \bigg|_{\mu} = \sigma_0^{ab}(\mu), \qquad F_{ab} \equiv \epsilon_{abc} B^c. \tag{22}$$

Such a Streda-type formula [14] should hold separately for each Fermi-surface sheet. This can easily be found from semiclassical quantization with a Fermi-surface Berry phase. For $\boldsymbol{B} = B\hat{\boldsymbol{n}}$ (technically parallel to a lattice translation), let the set of Fermi-surface orbits in the $\boldsymbol{k}$-space plane $\hat{\boldsymbol{n}} \cdot \boldsymbol{k} = k$ have cross-sectional areas $A_\alpha(k)$ and Berry phases $\phi_\alpha(k)$.

Recall that σ_0^{ab} is parametrized by $\boldsymbol{k}\alpha$: here

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{K}_\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk P_{\mathrm{BZ}}(k\hat{\boldsymbol{n}}) \phi_\alpha(k). \tag{23}$$

The semiclassical quantization condition is

$$A_\alpha(k_{F\alpha}^i)\ell_B^2 - \phi_\alpha(k_{F\alpha}^i) = 2\pi\left(n_\alpha^i + \frac{1}{2}\right), \tag{24}$$

where $\ell_B^2 = \hbar/|eB|$, and n_α^i are integers. The $k_{F\alpha}^i$ (ordered so $k_{F\alpha}^i > k_{F\alpha}^j$ for $i > j$) are the Fermi momenta of sets of 1D Fermi gases of particles on sheet α, moving along field lines with a 2D density $(2\pi\ell_B^2)^{-1}$ (one line per flux quantum). The induced charge density is completely determined by Fermi-surface geometry:

$$\rho_e = \frac{e}{(2\pi)^3} \sum_{\alpha i} \frac{1}{2} (A_\alpha^i + A_\alpha^{i+1}) \Delta k_\alpha^{i,i+1} - v_\alpha^{i,i+1}, \tag{25}$$

where $A_\alpha^i = A_\alpha(k_{F\alpha}^i) - (eB/\hbar)\phi_\alpha(k_{F\alpha}^i)$, $v_\alpha^{i,i+1}$ is the integral of $A_\alpha(k)$ from $k_{F\alpha}^i$ to $k_{F\alpha}^{i+1}$, and $\Delta k_\alpha^{i,i+1} = k_{F\alpha}^{i+1} - k_{F\alpha}^i$. The Streda formula (23) is now easily obtained when $B \to 0$. Note the differences between density and energy shifts due to Landau quantization: the latter derive from changes to states deep below the Fermi level, and Landau diamagnetism is not a Fermi-surface effect.

In summary, the intrinsic Hall conductivity of a metal with broken time-reversal symmetry can be written as $\sigma_0^{ab} = (e/\Phi_0)\epsilon^{abc}(K_c/2\pi)$, $\Phi_0 = h/e$. I have shown that the nonquantized part of the wave vector $\boldsymbol{K}$ (i.e., the part modulo a reciprocal vector $\boldsymbol{G}$) is a topological Fermi surface property given by a sum of terms $\boldsymbol{K}_\alpha$ (21) associated with each distinct Fermi-surface sheet. Separate batically conserved currents are associated with each such sheet (or group of sheets mutually coupled by "wormholes"): this generalizes the "extra" conservation laws at each distinct chiral Fermi point of a 1D Luttinger liquid. In the absence of BCS pairing processes, breakdown of these extra conservation laws can occur only through nonadiabatic impurity or surface scattering. A "topological Fermi liquid theory" that includes Berry phases of quasiparticles adiabatically moving on Fermi surface manifolds must now be developed.

This work was supported in part by the U.S. National Science Foundation (under MRSEC Grant No. DMR02-13706) at the Princeton Center for Complex Materials, and under Grant No. PHY99-07949 at the Kavli Institute for Theoretical Physics, UC Santa Barbara.

Note added.—The simplicity of the Fermi-surface formula (21) strongly suggests that it is a fundamental Fermi liquid property, also valid in an interacting system, like the relation between electron density and Fermi surface volume [used above to derive (22)]. As I will describe elsewhere, this is indeed the case: the key point is that (only) at the Fermi surface, the Bloch states $|\psi_n(\boldsymbol{k}_F(\boldsymbol{s}))\rangle$ retain their meaning as the zero-mode eigenfunctions of the inverse (exact) one-electron propagator $G^{-1}(\boldsymbol{k}_F(\boldsymbol{s}), \omega = 0)$, which in interacting Fermi-liquid theory remains Hermitian at $T = 0$. The formal proof uses a 3D version of the Ward-Takahashi identity used previously to relate the 2D integer QHE Hall conductance to the exact (interacting) one-electron propagator [15]. I thank P. B. Wiegmann for bringing Ref. [15] to my attention.

[1] R. Karplus and J. M. Luttinger, Phys. Rev. **95**, 1154 (1954); J. M. Luttinger, Phys. Rev. **112**, 739 (1958).
[2] T. Jungwirth, Q. Niu, and A. H. MacDonald, Phys. Rev. Lett. **88**, 207208 (2002).
[3] M. Onoda and N. Nagaosa, J. Phys. Soc. Jpn. **71**, 19 (2002).
[4] Z. Fang *et al.*, Science **302**, 92 (2003).
[5] Wei-Li Lee, Satoshi Watauchi, R. J. Cava, and N. P. Ong, Science **303**, 1647 (2004).
[6] M. V. Berry, Proc. R. Soc. London A **392**, 45 (1984).
[7] G. Sundaram and Q. Niu, Phys. Rev. B **59**, 14915 (1999).
[8] M. P. Marder, *Condensed Matter Physics* (Wiley, New York, 1999), pp. 425–429.
[9] M. Kohmoto, B. I. Halperin, and Y.-S. Wu, Phys. Rev. B **45**, 13488 (1992).
[10] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).
[11] J. E. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **51**, 51 (1983).
[12] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[13] Kramers degeneracy with spin-orbit coupling and inversion symmetry leads to non-Abelian SU(2) Berry curvature on the Fermi surface, to be described elsewhere.
[14] P. Streda, J. Phys. C **15**, L717 (1982).
[15] K. Ishikawa and T. Matsuyama, Z. Phys. C **33**, 41 (1986); Nucl. Phys. **B280**, 23 (1987).