# Robustness of the Noise-Induced Phase Synchronization in a General Class of Limit Cycle Oscillators

Jun-nosuke Teramae*
*Brain Science Research Center, Tamagawa University, Machida, Tokyo, Japan*

Dan Tanaka†
*Department of Physics, Graduate School of Sciences, Kyoto University, Kyoto 606-8502, Japan*
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We show that a wide class of uncoupled limit-cycle oscillators can be in-phase synchronized by common weak additive noise. An expression of the Lyapunov exponent is analytically derived to study the stability of the noise-driven synchronizing state. The result shows that such a synchronization can be achieved in a broad class of oscillators with little constraint on their intrinsic property. On the other hand, the leaky integrate-and-fire neuron oscillators do not belong to this class, generating intermittent phase slips according to a power law distribution of their intervals.

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Populations of nonlinear oscillators can be found in a variety of phenomena, including laser array [1], semiconductors [2], chemical reactions [3], society of living organisms [4], and neurons [5]. In many of these systems, the phases of oscillations can precisely coincide owing to mutual interactions among oscillators. Alternatively, a strong periodic input may synchronize independent oscillators through the entrainment to the common input. In either cases, external or internal noise sources may disturb the phase synchronization, and therefore have long been considered to exert a negative influence on the precise temporal relationship between oscillators.

This view, however, has been challenged recently. Pikovsky studied [6,7], in his pioneering work, a population of circle maps stimulated by impulse inputs at discrete random times and found that the common noise can induce stable phase synchronization. Since the noise-driven synchronization does not depend on the intrinsic frequency of oscillators, it differs from the entrainment to an external periodic input. Evidence is accumulating for the common-noise-induced synchronization in several biological and physical systems. For instance, an ensemble of independent neuronal oscillators may be synchronized by a fluctuating input applied commonly to all of them. This is suggested by experimental studies of neural information coding [8], in which the reproducibility of spike firing was tested for a repeated application of the same fluctuating or constant input current. Interestingly, the reproducibility of the output spike trains was much higher for the fluctuating input than for the constant one [8–10], indicating a high temporal precision of the spike responses to noisy input. In ecological systems, common environmental fluctuations such as climate changes may synchronize different populations separated by a large geographical distance [11]. In fluid dynamics, a common turbulent flow may generate a synchronized motion of floating particles [12].

All of these findings indicate an active role of noise in synchronization of noninteracting dynamical elements. It remains, however, unclear whether the noise-induced phase synchronization is specific to a limited class of oscillators or can be generalized to a broad class of oscillators. In this study of a general class of limit-cycle oscillators, we show that common additive noise, even if it is weak, can induce phase synchronization regardless of their intrinsic properties and the initial conditions. Using the phase reduction method which is applicable to an arbitrary oscillator [3], we analytically calculate the Lyapunov exponent of the synchronizing state and prove that the exponent is nonpositive as long as the phase-dependent sensitivity is differentiable up to the second order. In addition, we investigate the scaling laws that appear in the dynamics of the relative phase when the perfect phase synchronization is deteriorated by a discontinuous phase-dependent sensitivity or oscillator-specific noise sources.

Population of $N$ identical nonlinear oscillators driven by common additive noise are described as

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}(\boldsymbol{X}_i) + \boldsymbol{\xi}(t), \tag{1}$$

where $i = 1, \ldots, N$ and $\boldsymbol{\xi}(t)$ is a vector of Gaussian white noise. The elements of the vector are normalized as $\langle \xi_l(t) \rangle = 0$ and $\langle \xi_l(t)\xi_m(s) \rangle = 2D_{lm}\delta(t-s)$, where $\boldsymbol{D} = (D_{lm})$ is a variance matrix of the noise components. Because all the oscillators are identical and do not interact with one another, we can study the phase synchronization of the entire population in a reduced system of two oscillators. Regarding the common noise as a weak perturbation to the deterministic oscillators, the phase reduction method gives the following dynamical equations of the phases:

$$\dot{\phi}_i = \omega + \boldsymbol{Z}(\phi_i) \cdot \boldsymbol{\xi}, \tag{2}$$

where $\omega$ is an intrinsic frequency of the unperturbed oscillators. $\boldsymbol{Z}$ is the phase-dependent sensitivity defined as $\boldsymbol{Z}(\phi) = \mathrm{grad}_{\boldsymbol{X}}\phi|_{\boldsymbol{X}=\boldsymbol{X}_0(\phi)}$, where $\boldsymbol{X}_0(\phi)$ is the unperturbed limit-cycle solution determined by $\boldsymbol{F}(\boldsymbol{X})$. We assume that $\boldsymbol{Z}$ is differentiable at least to the second order, although $\boldsymbol{Z}$ can be discontinuous for such oscillators that have discontinuous periodic solutions (e.g., integrate-and-fire neurons). As we will see later, the discontinuity of $\boldsymbol{Z}$ can significantly affect the noise-driven synchronization. To ensure the validity of the phase reduction, the weak noise must satisfy the condition $|D_{lm}| \ll 1$.

Equation (2) implies that the synchronizing solution described as $\phi_1(t) = \phi_2(t)$ is absorbing; i.e., once two oscillators synchronize, they always remain synchronizing. Since the area of the phase space is limited ($0 \leq \phi_1, \phi_2 < 2\pi$), the phase variables starting from arbitrary initial phases can reach a neighborhood of the synchronizing solution with a finite probability in a finite time. To prove that the synchronizing solution is stable against perturbations, we analytically calculate the Lyapunov exponent $\lambda$ of the solution. We note that the stochastic Eq. (2) should be interpreted as a Stratonovich differential equation, since the phase reduction method assumes the conventional variable translations in differential equations. To evaluate the correlation between $\phi$ and $\boldsymbol{\xi}$, we translate Eq. (2) into an equivalent Ito differential equation [13]:

$$\dot{\phi}_i = \omega + \boldsymbol{Z}'(\phi_i)^T \boldsymbol{D}\boldsymbol{Z}(\phi_i) + \boldsymbol{Z}(\phi_i) \cdot \boldsymbol{\xi}, \tag{3}$$

where the dash denotes differentiation with respect to $\phi$. In the Ito equation, unlike in Stratonovich formulation, the correlation between $\phi$ and $\xi$ vanishes. The disappeared correlation is exactly compensated by the new extra drift term $\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z}$ in Eq. (3). Note that this translation is formal and mathematically equivalent requiring no approximation. Suppose that the two phases have an infinitesimally small difference $\psi = \phi_2 - \phi_1$. Then, linearization of Eq. (3) with respect to $\psi$ gives

$$\dot{\psi} = [(\boldsymbol{Z}'(\phi)^T \boldsymbol{D}\boldsymbol{Z}(\phi))' + \boldsymbol{Z}'(\phi) \cdot \boldsymbol{\xi}]\psi, \tag{4}$$

where $\phi$ obeys Eq. (3). By introducing a new variable $y = \log(\psi)$, Eq. (4) is further rewritten as

$$\dot{y} = (\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z})' - (\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z}') + \boldsymbol{Z}' \cdot \boldsymbol{\xi}. \tag{5}$$

Since the Lyapunov exponent is defined as $\lim_{T\to\infty}[y(T) - y(0)]/T$, the long time average of the right-hand side of Eq. (5) coincides with $\lambda$. Replacing the long time average with the ensemble average with respect to $\boldsymbol{\xi}$, we can represent $\lambda$ as

$$\begin{aligned}\lambda &= \langle(\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z})' - (\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z}') + \boldsymbol{Z}' \cdot \boldsymbol{\xi}\rangle_\xi \\ &= \langle(\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z})' - (\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z}')\rangle_\xi \\ &= \int_0^{2\pi} P_{\mathrm{st}}(\phi)[(\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z})' - (\boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z}')]d\phi. \end{aligned} \tag{6}$$

Here, the second line follows from $\langle \boldsymbol{Z}'(\phi) \cdot \boldsymbol{\xi}\rangle_\xi = 0$, which holds in Ito stochastic processes. $P_{\mathrm{st}}$ is a steady distribution function of $\phi$ described as $P_{\mathrm{st}}(\phi) = \frac{C}{|\boldsymbol{Z}(\phi)|^2} \times \int_\phi^{\phi+2\pi} \exp[V(x) - V(\phi)]dx$ with an effective potential $V(\phi) = -\int^\phi \frac{\omega + \boldsymbol{Z}'(x)^T\boldsymbol{D}\boldsymbol{Z}(x)}{\boldsymbol{Z}(x)^T\boldsymbol{D}\boldsymbol{Z}(x)}dx$ and a normalization constant $C$. Fortunately, under the assumption of weak noise, $|D_{lm}| \ll 1$, $P_{\mathrm{st}}$ is reduced to a constant function $P_{\mathrm{st}}(\phi) = 1/(2\pi)$. By substituting this into the last line of Eq. (6), and noting that the first term vanishes due to the periodicity of $\boldsymbol{Z}$, we finally obtain the following formula:

$$\lambda = -\frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{Z}'^T\boldsymbol{D}\boldsymbol{Z}' d\phi \leq 0, \tag{7}$$

where the equality holds if $\boldsymbol{Z}$ is a constant function. Since the variance matrix is positive definite, $\lambda$ is nonpositive. This implies that the phase synchronization induced by common noise is stable in an arbitrary oscillator system regardless of the detailed oscillatory dynamics, as long as $\boldsymbol{Z}$ is differentiable.

To confirm the above analytical results, we numerically solve Eq. (2) to obtain the Lyapunov exponent for a specific case, i.e., $\boldsymbol{Z}(\phi) \cdot \boldsymbol{\xi} = \sin(\phi)\xi$. This specific case is, for instance, given by Stuart-Landau oscillator described below with $c_0 = 1$, $c_2 = 0$, and $\boldsymbol{\xi}^T = (-\xi, 0)$. The phase difference between the two oscillators driven by common additive noise shows an exponential decay, and the decay constant well agrees with the analytical result [Fig. 1(a)]. Consistent with Eq. (7), the magnitude of the negative Lyapunov exponent increases in proportion to the noise intensity $D$ [Fig. 1(b)]. In order to confirm the validity of the phase reduction method, we employ the Stuart-Landau oscillators and compare the Lyapunov exponent derived from Eq. (7) with that calculated numerically in the original oscillator system [Fig. 1(c)]. For the Stuart-Landau oscillator described with a complex variable $A$ as $\dot{A} = (1 + ic_0)A - (1 + ic_2)|A|^2A$, the phase sensitivity can be explicitly given as $\boldsymbol{Z}(\phi)^T = (-c_2\cos\omega\phi - \sin\omega\phi, -c_2\sin\omega\phi + \cos\omega\phi)/\omega$, where $\omega = c_0 - c_2$ [3].

In practical situations, the individual oscillators may be influenced by additional oscillator-specific noise sources. To discuss the influences of additional noise, Eq. (1) is modified to

$$\dot{\boldsymbol{X}}_i = \boldsymbol{F}(\boldsymbol{X}_i) + \boldsymbol{\xi}(t) + \boldsymbol{\eta}_i(t), \tag{8}$$

where the uncommon noise sources $\boldsymbol{\eta}_i$ are normalized as $\langle\eta_{i,l}(t)\rangle = 0$ and $\langle\eta_{i,l}(t)\eta_{j,m}(s)\rangle = 2d_{lm}\delta_{ij}\delta(t-s)$. Linearization of Eq. (8) gives the stochastic equation of the phase difference as

$$\begin{aligned}\dot{\psi} = &\left[(\boldsymbol{Z}'^T(\boldsymbol{D}+\boldsymbol{d})\boldsymbol{Z})' + \boldsymbol{Z}' \cdot \left(\boldsymbol{\xi} + \frac{\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2}{2}\right)\right]\psi \\ &+ \boldsymbol{Z} \cdot (\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1). \end{aligned} \tag{9}$$

Since both multiplicative and additive factors fluctuate, Eq. (9) is regarded as a multiplicative stochastic process with additive noise, which has been studied in a variety of

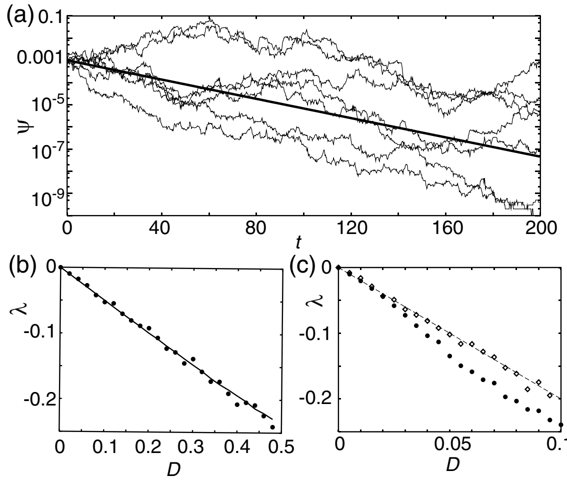


FIG. 1. The common-noise-induced phase synchronization. (a) The phase difference between two oscillators shows an exponential decay fluctuating around the theoretical behavior (solid line) derived from Eq. (7). Different curves correspond to different realizations of the random driving force $\xi(t)$. Here, $\mathbf{Z}\cdot\boldsymbol{\xi} = \sin(\phi)\xi$, $\omega = 1$, and $D = 0.1$. (b) The Lyapunov exponent of the above synchronizing solution is shown as a function of the common-noise intensity (circles). The solid line shows an analytical result. (c) The Lyapunov exponent is calculated for the synchronizing solution of the Stuart-Landau oscillator. Numerical results are shown for both original (solid circles) and reduced (open squares) oscillator systems. The solid line represents a theoretical result. The parameter values are set as $c_0 = 2$, $c_2 = 1$, $D_{11} = D_{22} = D$, and $D_{12} = D_{21} = 0$.

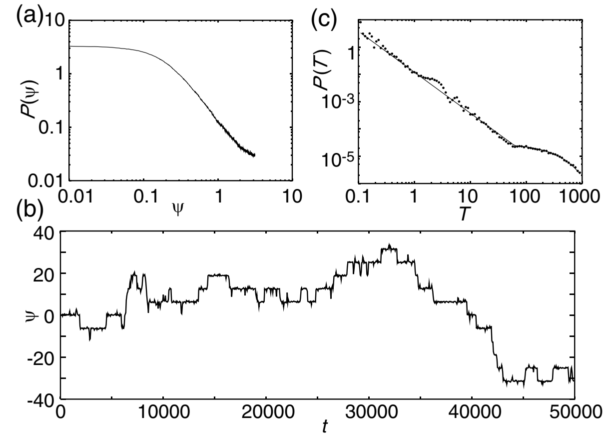


FIG. 2. The intermittent phase slips induced by uncommon additive noise sources to the oscillators defined by $\omega = 1$, $\mathbf{Z}\cdot\boldsymbol{\xi} = \sin(\phi)\xi$, and $\mathbf{Z}\cdot\boldsymbol{\eta} = \sin(\phi)\eta$. The noise intensities are $D = 0.1$ and $d = 0.001$. (a) The distribution of the phase difference over a sufficiently long time. (b) The time evolution of the phase difference exhibits intermittent phase slips. (c) The distribution of the inter-phase-slip intervals shows a power law decay with an exponent of $-3/2$ (fitted by a solid line) and an exponential cutoff resulted from the additive noise term in Eq. (9).

fields [14]. The steady distribution function of Eq. (9) exhibits a power law decay in a middle range of $\psi$ [Fig. 2(a)]. Simulations of Eq. (8) reveal that the system is trapped in the phase synchronizing state for certain intervals between intermittent phase slips [Fig. 2(b)]. It is known that the intermittent bursts are characteristic to the stochastic processes driven simultaneously by multiplicative and additive noise sources [15], and that the intervals between neighboring bursts obey a power law distribution with an exponent of $-3/2$. In Fig. 2(c), the interslip intervals of the present phase dynamics obey a power law distribution of the same exponent.

So far, we have assumed that the phase-dependent sensitivity $\mathbf{Z}$ is a continuous function of the phase. However, some oscillators do not have this property. For example, an integrate-and-fire neuron oscillator, which is described by $\dot{v} = I - v$ with a renewal condition $v(t) = 1 \rightarrow \lim_{\tau\rightarrow 0+} v(t+\tau) = 0$, is frequently used for modeling neuronal activity, but it has the following discontinuous $\mathbf{Z}$:

$$Z(\phi) = \frac{\omega}{I}\exp\left(\frac{\phi}{\omega}\right), \qquad 0 \le \phi < 2\pi, \tag{10}$$

where $\omega = 2\pi/[\log I - \log(I-1)]$ [16]. As shown in Fig. 3(a), numerical integrations of Eq. (2) for the $\mathbf{Z}$ given in Eq. (10) show positive Lyapunov exponents, with the magnitudes increased with the intensity of the common noise. The phase difference does not decay exponentially but fluctuates around the synchronizing state between the intermittent phase slips [Fig. 3(b)]. Figure 3(c) displays the synchronized time evolution of the phase variables that is terminated by an abrupt phase slip at $t \approx 170$: After that, the two phases are desynchronized until they recover the phase synchronization (not shown). As in the previous case, the interslip intervals obey a $-3/2$ power law distribution [Fig. 3(d)]. This intermittency is essentially the same as the on-off intermittency of chaotic oscillators just before the onset of synchronization, thus associated with positive Lyapunov exponents [17]. Note that the discontinuity of the phase sensitivity and the resultant positive Lyapunov exponents are inherent in the leaky integrate-and-fire model. For example, the Hodgkin-Huxley model has a continuous and differentiable phase sensitivity, thus yielding a negative Lyapunov exponent and a stable phase synchronization in response to common noise (results not shown).

We briefly argue the relationships between the present study and two previous studies. In the stochastic resonance, the ability of an excitable system in detecting a weak signal can be optimized by noise of suitable intensity [18]. The present study also argues the role of noise in improving the response reliability. However, here the improvement is achieved by the precise temporal coincidences between oscillators, whereas the stochastic resonance enhances only the response probability without caring the exact timing of events. Thus, the stability or Lyapunov exponent is not a central issue in the stochastic resonance, and the two studies deal with qualitatively different phenomena. Second, some chaotic oscillators

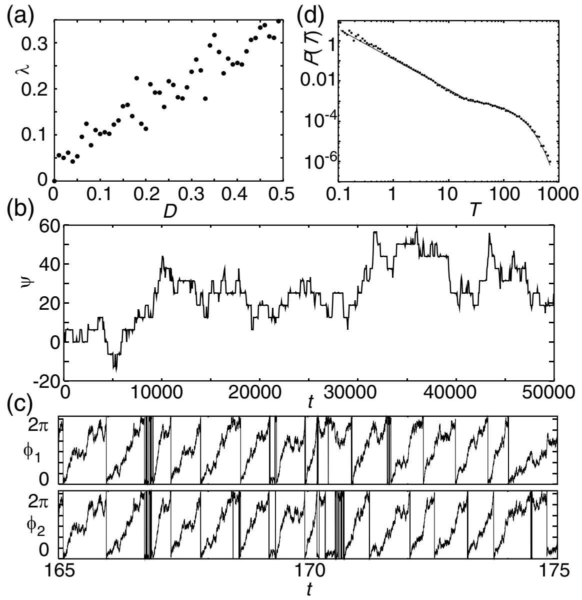

FIG. 3. The unstable phase synchronization of integrate-and-fire models stimulated by a common-noise source. The phase-dependent sensitivity of this oscillator is discontinuous, and the noise-driven synchronization is not ensured. $I = 2$ and no uncommon noise, $d = 0$. (a) The Lyapunov exponent $\lambda$ is plotted as a function of the common-noise strength $D$. (b) The phase difference shows significant jitters, diffusing due to intermittent phase slips. Here and below, $D = 0.1$. (c) The time evolution of the two phases displays temporarily synchronizing ($t < 170$) and desynchronizing ($t > 170$) states. (d) The distribution of the inter-phase-slip intervals shows an exponential decay with an exponent of $-3/2$ (fitted by a solid line).

exhibited phase synchronization when they were driven by common additive noise [7,19]. However, it remained unknown whether this type of synchronization may appear in a broad class of, either chaotic or nonchaotic, oscillator systems. In this Letter, we have proven that such a synchronization can be induced in a wide class of limit-cycle oscillators. A unified treatment of limit-cycle oscillators and chaotic oscillators is awaited, as they may share many characteristic properties of the common-noise-induced synchronization.

In conclusion, independent limit-cycle oscillators can be synchronized by weak, common additive noise regardless of the detailed oscillatory dynamics and the initial phase distributions. The stability of this synchronizing solution requires only the presence of a second derivative of the phase-dependent sensitivity, so the solution can exist in a broad class of oscillators. The leaky integrate-and-fire oscillators do not belong to this class of oscillators and show no perfect synchronization.

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*Electronic address: teramae@brain.inf.eng.tamagawa.ac.jp
†Electronic address: dan@ton.scphys.kyoto-u.ac.jp

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