Classical Physics and Quantum Loops

Barry R. Holstein[1,2] and John F. Donoghue[1]

[1]*Department of Physics-LGRT, University of Massachusetts, Amherst, MA 01003, USA*

[2]*Theory Group, Thomas Jefferson National Accelerator Laboratory, 12000 Jefferson Avenue, Newport News, VA 23606, USA*

(Received 13 July 2004; published 11 November 2004)

The standard picture of the loop expansion associates a factor of $\hbar$ with each loop, suggesting that the tree diagrams are to be associated with classical physics, while loop effects are quantum mechanical in nature. We discuss counterexamples wherein classical effects arise *from loop diagrams* and display the relationship between the classical terms and the long range effects of massless particles.

DOI: 10.1103/PhysRevLett.93.201602 PACS numbers: 11.10.–z, 03.65.Sq

It is commonly stated that the loop expansion in quantum field theory is equivalent to an expansion in $\hbar$ [1]. Although this is mentioned in several field theory textbooks, we have not found a fully compelling proof of this statement. Indeed, no compelling proof is possible because the statement is not true in general. In this Letter we describe several exceptions, cases where classical effects are found within one-loop diagrams, and discuss what goes wrong with purported "proofs."

Most physicists performing quantum mechanical calculations eschew keeping track of factors of $\hbar$, and use units wherein $\hbar$ is set to unity—only when numerical results are needed are these factors restored. However, use of this procedure can cloak the difference between classical and quantum mechanical effects, since the former are distinguished from the latter merely by the absence of factors of $\hbar$. This is also the practice in many field theory texts, but there is often a discussion in such works about a one to one connection between the number of loops and the factors of $\hbar$ [1]. The argument used in order to make this connection is a simple one, and is worth outlining here: in calculating a typical Feynman diagram, the presence of a vertex arises from the expansion of

$$\exp\frac{i}{\hbar}\int \mathcal{L}_{\rm int}(\phi_{\rm in})d^4x,$$

and so carries with it a factor of $\hbar^{-1}$. On the other hand, the field commutation relations $[\phi(\vec{x}), \pi(\vec{y})] = i\hbar\delta^3(\vec{x}-\vec{y})$ lead to a factor of $\hbar$ in each propagator

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \int\frac{d^4k}{(2\pi)^4}\frac{i\hbar e^{ik(x-y)}}{k^2-\frac{m^2}{\hbar^2}-i\epsilon}.$$

The counting of factors of $\hbar$ then involves calculating the number of vertices and propagators in a given diagram. For a diagram with V vertices and I internal lines, the number of independent momenta is $L = I - V + 1$ and corresponds to the number of loops. Associating a factor of $\hbar^{-1}$ for the V vertices and $\hbar^{+1}$ for the I propagators yields an overall factor $\hbar^{I-V+1} = \hbar^L$, which is the origin of the claim that the loop expansion coincides with an expansion in $\hbar$. We shall demonstrate, however, that this assertion in not valid.

Let us give an example where one easily identifies classical results from a one-loop calculation. We describe the one-loop QED calculation of the matrix element of the energy-momentum tensor between initial and final plane wave states [2]. The basic structure of this matrix element is given by

$$\langle p_2|T_{\mu\nu}(0)|p_1\rangle = \frac{1}{\sqrt{4E_2E_1}}[2P_\mu P_\nu F_1(q^2) + (q_\mu q_\nu - \eta_{\mu\nu}q^2)F_2(q^2)], \quad (1)$$

where $F_1(q^2), F_2(q^2)$ are form factors to be determined. In lowest order, the energy-momentum tensor form factors are $F_1(q^2) = 1$, $F_2(q^2) = -1/2$. The form factors will receive corrections of order e^2 at one-loop order via a straightforward calculation [2]. The results are

$$F_1(q^2) = 1 + \frac{e^2}{16\pi^2}\frac{q^2}{m^2}\left(\frac{3}{4}\frac{m\pi^2}{\sqrt{-q^2}} - \frac{8}{3} + 2\log\frac{-q^2}{m^2} - \frac{4}{3}\log\frac{\lambda}{m}\right) + \ldots$$

$$F_2(q^2) = -\frac{1}{2} + \frac{e^2}{16\pi^2}\left(\frac{m\pi^2}{2\sqrt{-q^2}} - \Omega - \frac{26}{9} + \frac{4}{3}\log\frac{-q^2}{m^2}\right) + \ldots \quad (2)$$

where $\Omega = 2/\epsilon - \gamma - \log(m^2/4\pi\mu^2)$.

It is easiest to separate the classical and quantum effects by going to coordinate space via a Fourier transform. The key terms are those that have a nonanalytic structure such as $\sqrt{-q^2/m^2}$ and $q^2\ln -q^2$. These both arise only from those diagrams where the energy-momentum tensor couples to the photon lines. In particular, the square root term comes uniquely from a diagram represented by Fig. 1, which will play an important role in all classical effects described in this Letter. We will see that the square root turns into a well-known classical correction while the logarithm generates a quantum cor-

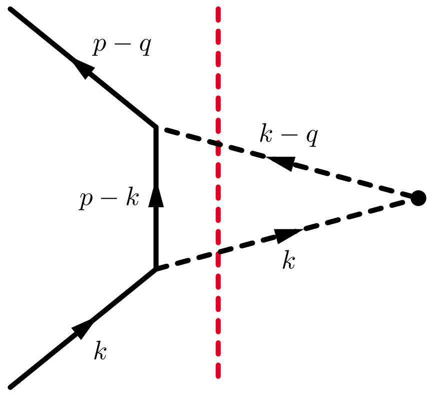

FIG. 1 (color online). Generic triangle diagram with two massless particles exchanged. In the first calculation, the point is the energy-momentum tensor coupling. In the polarizability calculation, it represents the effective vertex for the polarizability. Finally, the dotted line describes the cut appropriate for the dispersive treatment later in the Letter.

rection. Specifically, we take the Fourier transform of the amplitude to yield (including here powers of $\hbar$)

$$T_{00}(\vec{r}) = m\delta^3(\vec{r}) + \frac{e^2}{32\pi^2 r^4} - \frac{e^2\hbar}{4\pi^3 m r^5} + \ldots$$
$$T_{0i}(\vec{r}) = 0$$
$$T_{ij}(\vec{r}) = -\frac{e^2}{16\pi^2 r^4}\left(\frac{r_i r_j}{r^2} - \frac{1}{2}\delta_{ij}\right) - \frac{e^2\hbar}{16\pi^3 m r^5}\left(3\delta_{ij} - 5\frac{r_i r_j}{r^2}\right) + \ldots \tag{3}$$

We see then that Eq. (3) includes both corrections which are independent of $\hbar$ as well as pieces that are linear in this quantity.

The interpretation of the classical terms is clear. Since the energy-momentum tensor for the electromagnetic field has the form [3]

$$T^{\rm EM}_{\mu\nu} = -F_{\mu\lambda}F^{\lambda}_{\nu} + \frac{1}{4}\eta_{\mu\nu}F_{\lambda\delta}F^{\lambda\delta}, \tag{4}$$

we determine the energy momentum due to the electrical classical field to be

$$T^{\rm EM}_{00}(\vec{r}) = \frac{1}{2}E^2 = \frac{e^2}{32\pi^2 r^4}; \qquad T^{\rm EM}_{0i}(\vec{r}) = 0$$
$$T^{\rm EM}_{ij}(\vec{r}) = -E_iE_j + \frac{1}{2}\delta_{ij}E^2 = -\frac{e^2}{16\pi^2 r^4}\left(\frac{r_i r_j}{r^2} - \frac{1}{2}\delta_{ij}\right), \tag{5}$$

which agree exactly with the components of Eq. (3) that fall as $1/r^4$. Despite arising from a loop calculation then, this is a *classical* effect, due to the feature that the energy-momentum tensor can couple to the electric field surrounding the particle as well as to the particle directly. At tree level, the energy-momentum tensor represents only that of the charged particle itself. However, the charged particle has an associated classical electric field and that field also carries energy momentum. From this point of view, it is not surprising that the calculation yields a classical term; there is energy in the classical field at this order in e^2, and a calculation at order e^2 must be capable of uncovering it. Of course, the full loop calculation also contains additional physics, the leading piece of which is quantum mechanical in nature and falls as $\hbar/mr^5$. So we see that the one-loop diagram contains both classical and quantum physics.

The argument that the loop expansion is equivalent to an expansion in $\hbar$ clearly failed in the above calculation. Let us look at this failure in more detail. One loophole to the original argument is visible in the propagator, which contains $\hbar$ in more than one location. When the propagator is written in terms of an integral over the wave number, the mass carries an inverse factor of $\hbar$. This is because the Klein-Gordon equation reads

$$\left(\Box + \frac{m^2}{\hbar^2}\right)\phi(x) = 0,$$

when $\hbar$ is made visible. This means that the counting of $\hbar$ from the vertices and the propagator is incomplete. One also needs to know how the mass enters the result, because there are factors of $\hbar$ attached therein also.

In the previously discussed loop calculation of the form factors of the energy-momentum tensor, we can display the factors of $\hbar$ in momentum space. Returning $\hbar$ to the formula for F_1 we find (we continue to use $c = 1$)

$$F_1(k) = 1 + \frac{e^2}{16\pi^2\hbar}\frac{\hbar^2k^2}{m^2}\left(\frac{3}{4}\frac{m\pi^2}{\sqrt{-\hbar^2k^2}} - \frac{8}{3} + 2\log\frac{-\hbar^2k^2}{m^2} - \frac{4}{3}\log\frac{\lambda}{m}\right) + \ldots$$
$$= 1 + \frac{3e^2\sqrt{-k^2}}{64m} + \frac{\hbar e^2k^2}{16\pi^2m^2}\left(-\frac{8}{3} + 2\log\frac{-\hbar^2k^2}{m^2} - \frac{4}{3}\log\frac{\lambda}{m}\right) + \ldots \tag{6}$$

Here we have written the momentum in terms of the wave number $q = \hbar k$, and we note that $e^2/\hbar$ is dimensionless in Gaussian units (with $c = 1$). It is easy to see, then, that the coefficient of the square root nonanalytic behavior is independent of $\hbar$, while the logarithmic term has one power of $\hbar$ remaining. The one-loop result carries different powers of $\hbar$ because it contains different powers of the factor q^2/m^2. Moreover, we can be more precise. With the general expectation of one factor of $\hbar$ at one-loop, there is a specific combination of the mass and momentum that eliminates $\hbar$ in order to produce a classical result. In order to remove one power of $\hbar$, one requires a factor of

$$\sqrt{\frac{m^2}{-q^2}} = \frac{m}{\hbar\sqrt{-k^2}}. \tag{7}$$

This is a nonanalytic term that is generated only by the propagation of massless particles. The emergence of the power of $\hbar^{-1}$ involves an interplay between the massive particle (whose mass carries the factor of $\hbar$) and the massless one (which generates the required nonanalytic form) [4]. This result suggests that one can generate classical results from one-loop processes in the presence of massless particles, which have long range propagation and therefore generate the required nonanalytic momentum behavior.

There are also other examples where classical results are found in one-loop calculations. All involve couplings to massless particles. Let us briefly describe some of these.

The calculation of the energy-momentum tensor can be extended to include graviton loops as has been done in Ref. [5]. Here there exists a superficial difference in that the gravitational coupling constant carries a mass dimension and the one-loop result involves the Newtonian gravitational constant G_N. This feature might be thought to change the counting in $\hbar$, but it does not. Again, the important diagrams are those in which the energy-momentum vertex couples to the graviton line. The calculation is similar to the above example and we refer the reader to [5] for details. The classical results emerge from a square root nonanalytic amplitude such as $F_1 = 1 + G\pi m\sqrt{-q^2}/16 + \ldots$, and are shown to both describe the energy contained in the gravitational field and to reproduce the classical nonlinear behavior of the Schwarzschild metric.

Another example from electromagnetism involves the interaction between an electric charge and a neutral system described by an electric polarizability. The classical physics here is clear: the presence of an electric charge produces an electric dipole moment $\vec{p}$ in the charge distribution of the neutral system, the size of which is given in terms of the electric polarizability α_E via

$$\vec{p} = 4\pi\alpha_E\vec{E}. \tag{8}$$

However, a dipole also interacts with the field, via the energy

$$U = -\frac{1}{2}\vec{p}\cdot\vec{E} = -\frac{1}{2}4\pi\alpha_E\vec{E}^2. \tag{9}$$

Since, for a point charge $\vec{E} = \frac{e\hat{r}}{4\pi r^2}$, there exists a simple classical energy

$$U = -\frac{1}{2}\frac{e^2\alpha_E}{4\pi r^4}. \tag{10}$$

This result can be obtained via a one-loop calculation. Again, for simplicity, we assume that both systems are spinless. The two-photon vertex associated with the electric polarizability can be modeled in terms of a transition to a $J^P = 1^-$ intermediate state, yielding the Compton structure

$$\mathrm{Amp}_E = \frac{8\pi}{m}\alpha_E[\epsilon_1\cdot\epsilon_2 P\cdot k_2 P\cdot k_1 + \epsilon_1\cdot P\epsilon_2\cdot Pk_1\cdot k_2 - \epsilon_1\cdot P\epsilon_2\cdot k_1 P\cdot k_2 - \epsilon_2\cdot P\epsilon_1\cdot k_2 P\cdot k_1], \tag{11}$$

where $P = \frac{1}{2}(p_1 + p_2)$ is the mean four-momentum. Calculating the effect of this term on the interaction energy of the neutral system and a charged particle involves the exchange of two photons. Performing the calculation, one finds the threshold amplitude in momentum space

$$\mathrm{Amp} = \frac{e^2q^2m}{4\pi}\left[2\alpha_E\pi^2\sqrt{\frac{M^2}{-q^2}} + \frac{11}{3}\alpha_E\log(-q^2)\right]. \tag{12}$$

Including the normalization factor $1/4mM$ and Fourier transforming, we find the potential energy

$$V_{2\gamma}(r) = -\frac{1}{2}\frac{e^2\alpha_E}{4\pi r^4} + \frac{11e^2\alpha_E\hbar}{4\pi M r^5} + \ldots \tag{13}$$

We see again that the one-loop calculation has yielded the classical term accompanied by quantum corrections. It should be noted here that, although we have represented the two-photon electric polarizability coupling in terms of a simple contact interaction as done by Bernabeu and Tarrach [6], the result is in complete agreement with a full box plus triangle diagram calculation by Sucher and Feinberg [7].

A similar result is obtained by considering the generation of an electric quadrupole moment by an external field gradient. Let us define the field gradient via $E_{ij} = \frac{1}{2}\times(\nabla_i E_j + \nabla_j E_i)$ and the quadrupole polarizability via $Q_{ij} = 4\pi\alpha_{E2}E_{ij}$. The classical energy due to interaction of this moment with the field gradient is given by

$$U = -\frac{1}{2}\alpha_{E2}E_{ij}E_{ij}. \tag{14}$$

The quadrupole polarizability can be modeled in terms of excitation to a $J^P = 2^+$ excited state and again, a simple one-loop calculation finds a combination of classical and quantum terms. Similarly, in a gravitational analog, the presence of a point mass produces a field gradient that generates a gravitational quadrupole, which in turn interacts with the field gradient and leads to a classical energy.

Finally, the gravitational potential between two heavy masses has been treated to one-loop in an effective field theory treatment of quantum gravity [8]. Again, the diagrams involving two graviton propagators in a loop yield square root nonanalytic terms that reproduce the nonlinear classical corrections to the potential that are predicted by general relativity [9]. This feature has been known for some time [10].

The lesson here is clear: these examples all involve one-loop diagrams that contain a combination of classical *and* quantum mechanical effects, wherein the classical piece is signaled by the presence of a square root nonanalyticity.

We can further understand the association of classical effects with massless particles by studying a dispersive treatment. In this approach, we can see directly that the classical terms are associated with the dispersion integral extending down to zero momentum, which is possible only if the particles in the associated cut are massless. It is useful to use the Cutkosky rules to look at the absorptive component of the triangle diagram shown in Fig. 1, wherein we assume (temporarily) that the exchanged particles have mass μ. A simple calculation yields [11]

$$\begin{aligned}\gamma(q^2) &\equiv \mathrm{Abs}\int\frac{d^4k}{(2\pi)^4}\\ &\quad\times\frac{1}{(k^2-\mu^2)[(k-q)^2-\mu^2][(k-p)^2-M^2]}\\ &=\int\frac{d^4k}{(2\pi)^4}\\ &\quad\times\frac{(2\pi i)^2\delta(k^2-\mu^2)\delta[(k-q)^2-\mu^2]}{(k^2-\mu^2)[(k-q)^2-\mu^2][(k-p)^2-M^2]},\end{aligned} \tag{15}$$

where

$$\gamma(q^2)=\frac{1}{8\pi\sqrt{q^2(4M^2-q^2)}}\tan^{-1}\frac{\sqrt{(q^2-4\mu^2)(4M^2-q^2)}}{q^2-2\mu^2}. \tag{16}$$

The corresponding dispersion integral is given by

$$\Gamma(q^2)=\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{dt\gamma(t)}{t-q^2-i\epsilon}. \tag{17}$$

The argument of the arctangent vanishes at threshold and the dispersion integral yields a form of no particular interest. On the other hand, in the limit $\mu\to 0$, the argument of the arctangent becomes infinite at threshold and instead we write

$$\gamma(q^2)=\frac{1}{8\pi\sqrt{q^2(4M^2-q^2)}}\left[\frac{\pi}{2}-\tan^{-1}\sqrt{\frac{q^2}{(4M^2-q^2)}}\right], \tag{18}$$

where we have separated the result into two components—the piece proportional to $\pi/2$, which arises from the on-shell (delta function) piece of the mass M propagator and the remaining terms that arise from the principal value integration. The dispersion integral now begins at zero and yields a logarithmic result from pieces of $\gamma(q^2)$, which behave as a constant as $q^2\to 0$, while square root pieces arise from terms in $\gamma(q^2)$ which behave as $1/\sqrt{q^2}$ in the infrared limit. From Eq. (18) we see that the former—the quantum component—arises from the principal value integration while the latter—the classical component—is associated with the on-shell contributions to $\gamma(q^2)$. This is to be expected. A classical contribution should arise from the case where both initial/final *and* intermediate state particles are on-shell and therefore physical.

We have seen above that in the presence of at least two massless propagators, classical physics can arise from loop contributions, in apparent contradiction to the usual loop-$\hbar$ expansion arguments. The presence of classical corrections are associated with a specific nonanalytic term in momentum space. Using a dispersion integral, the origin of this phenomenon has been traced to the infrared behavior of the Feynman diagrams involved, which is altered dramatically when the threshold of the dispersion integration is allowed to vanish, as can occur when two or more massless propagators are present. We conclude that the standard expectation that the loop expansion is equivalent to an $\hbar$ expansion is not valid in the presence of coupling to two or more massless particles.

We thank A. Zee for comments on the manuscript. This work was supported in part by the National Science Foundation under Grant No. PHY-02-44801.

[1] See, e.g., R. J. Rivers, *Path Integral Methods in Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1987), Sec. 4.6; L. H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1985), p. 327; C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), Sec. 6-2-1.

[2] J. F. Donoghue, B. R. Holstein, B. Garbrecht, and T. Konstandin, Phys. Lett. B **529**, 132 (2002).

[3] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

[4] From this counting, it is also clear that in a derivative/momentum expansion, higher powers of q^2/m^2 can lead to higher powers of $\hbar$.

[5] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, Phys. Rev. D **68**, 084005 (2003).

[6] J. Bernabeu and R. Tarrach, Ann. Phys. (N.Y.) **102**, 323 (1976).

[7] G. Feinberg and J. Sucher, Phys. Rev. A **27**, 1958 (1983).

[8] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, Phys. Rev. D **67**, 084033 (2003).

[9] J. F. Donoghue, Phys. Rev. D **50**, 3874 (1994); I. B. Khriplovich and G. G. Kirilin, gr-qc/0402018; JETP **95**, 981 (2002).

[10] S. N. Gupta and S. F. Radford, Phys. Rev. D **21**, 2213 (1980); D. G. Boulware and S. Deser, Ann. Phys. (N.Y.) **89**, 193 (1975).

[11] S. Scherer, Adv. Nucl. Phys. **27**, 277 (2003).