Operational Representation of Quantum States Based on Interference

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We describe a real-valued and periodic representation of quantum states. This representation can be defined operationally using generalized position and momentum measurements on coupled systems. It turns out that the emerging quantum interference terms encode the complete state information and also allow us to formulate quantum dynamics. We discuss the close connection to the theory of analytic functions.

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The interpretation of a quantum state is a persistent problem since the early days of modern quantum mechanics [1]. Via the Born rule [2] it can be used to calculate probabilities of measurement results. On this level a quantum state basically serves as a technical tool of the theory. The corresponding encoding in terms of a Hilbert space vector $|\psi\rangle$, or in general, as a density operator $\hat{\rho}$ is a mathematically elegant and consistent way to represent the complete knowledge about a quantum system. This means that it successfully connects the macroscopic actions undertaken for the preparation of a system to the possible macroscopic observations [3]. The state is, however, not a directly measurable quantity and therefore has no immediate operational meaning.

Therefore, the concept of a quantum state differs crucially from the concept of a classical state. In each classical theory of physics, the corresponding state of a system always consists of a sufficient collection of measurable quantities. Positions and momenta in classical mechanics are the most prominent examples. As a consequence any classical state has an operational meaning. It is then rather straightforward to formulate a realistic interpretation for classical theories in physics.

On the other hand quantum mechanics builds upon a distinction between the state $|\psi\rangle$ of a system and an observable $\hat{A}$, which relates to the macroscopic signatures of the system. The connection is made via the representation $\langle a|\psi\rangle$ defined by the scalar product with the eigenstates $|a\rangle$ of the Hermitian operator $\hat{A}$. The complex-valued amplitude $\langle a|\psi\rangle$ is not a measurable quantity. The corresponding probability distribution $|\langle a|\psi\rangle|^2$ for the eigenvalues has an operational meaning, but in general it does not allow us to uniquely identify the state $|\psi\rangle$.

In analogy to the classical case of positions and momenta one might suspect that the probability distributions of complementary observables could be sufficient to determine $|\psi\rangle$. However, well-known counterexamples can be formulated already for massive particle in one dimension: its position distribution $|\langle x|\psi\rangle|^2$ and the complementary momentum distribution $|\langle p|\psi\rangle|^2$ do not [4] uniquely represent the underlying state $|\psi\rangle$. One rather needs [5,6] the set of all tomographic distributions $|\langle X_\theta|\psi\rangle|^2$ for the observable $\hat{X}_\theta \equiv \hat{x}\cos\theta + \hat{p}\sin\theta$ parametrized [7] by the angle $\theta \in [0, \pi]$. Hence the complete state information is here represented by an infinite ensemble of *continuous* probability distributions.

One might argue that the quantum state contains all potentialities that are macroscopically realized in the interaction with a suitable measuring apparatus. Therefore, we now pose the question of what type of a conceptually simple measurement on a quantum particle brings out the complete state information and provides an operational representation of the state $|\psi\rangle$. In fact we will show that such a measurement can be formulated on the basis of generalized position and momentum observables.

Guided by the concept of a classical measurement we compare the unknown particle to a known quantum ruler [8]. This quantum ruler is assumed to be a calibrated system for which we know all properties. Comparison means that we scan positions and momenta of the quantum particle relative to a set of known states $|\varphi\rangle$ of the ruler. This can be done [8,9] using the jointly measurable observables $\hat{X} = \hat{x}_0 - \hat{x}_1$ and $\hat{P} = \hat{p}_0 + \hat{p}_1$. Here the index 0 refers to the ruler and index 1 denotes the system to be measured. The corresponding simultaneous eigenstates $|X, P\rangle$ for the pair (X, P) of eigenvalues allow us to define the probability

$$W \equiv |\langle X = 0, P = 0|\varphi\rangle|\psi\rangle|^2 \qquad (1)$$

to find agreement in the positions and opposite momenta of both systems [8]. This clearly corresponds to a comparison of the quantum ruler to the unknown system [10]. Actually one can immediately see that this operational quantity W contains the complete information on the state $|\psi\rangle\langle\psi|$ [11] for the coherent-state ruler $|\varphi\rangle = |\alpha^*\rangle$. In this case Eq. (1) reduces [8,12] to the Husimi function $\frac{1}{\pi}|\langle\alpha|\psi\rangle|^2$ which clearly determines $|\psi\rangle\langle\psi|$ if measured for *all* parameters α in the complex plane [13,14].

However, the question arises of whether a more suitable quantum ruler could be used. In particular, we would like to exploit its quantum features. Therefore, we propose an alternative comparison of system and ruler which is based on quantum interference. The corresponding quantum ruler is supposed to be in a superposition state $|\varphi\rangle = \mathcal{N}(|\alpha^*\rangle + e^{i\gamma}|\beta^*\rangle)$ of two coherent states with a controllable phase γ and normalization $|\mathcal{N}|^{-2} \equiv 2 + 2\mathrm{Re}\{\langle\alpha^*|\beta^*\rangle e^{i\gamma}\}$. In this case the operational probability, Eq. (1), reads

$$W = \frac{|\mathcal{N}|^2}{\pi}[|\langle\alpha|\psi\rangle|^2 + |\langle\beta|\psi\rangle|^2 + 2\mathrm{Re}\{e^{i\gamma}\langle\beta|\psi\rangle\langle\psi|\alpha\rangle\}]. \tag{2}$$

In fact we will show that the quantum interference part of this probability contains the full information on the state $|\psi\rangle\langle\psi|$ in an elegant way. First we can isolate the pure interference term by determining the probabilities, Eq. (2), for the phases $\gamma = 0$ and $\gamma = \pi$. Hence we can operationally define the u function

$$u(\alpha) \equiv e^{|\alpha|^2/2}\mathrm{Re}\{\langle\beta|\psi\rangle\langle\psi|\alpha\rangle\}; \qquad \alpha \in \text{closed curve}, \tag{3}$$

where we assume the complex constant $\langle\beta|\psi\rangle$ to be different from zero [15]. This u function, that is the real-valued interference part of the joint position-momentum probability, Equation (2), on an arbitrary closed curve in the complex α plane contains the complete state $|\psi\rangle\langle\psi|$. We emphasize that this corresponds to a real-valued, one-parametric representation which, in addition, is periodic. Hence the representation of $|\psi\rangle\langle\psi|$ via the u function turns out to be much simpler than representing it with the help of any real-valued quasiprobability distribution.

In order to prove this completeness of the u function, Eq. (3), we recall the expansion of coherent states $|\alpha\rangle = e^{-(1/2)|\alpha|^2}\sum_{n=0}^{\infty}(\alpha^n/\sqrt{n!})|n\rangle$ in terms of Fock states $|n\rangle$ and notice that it represents the real part of the completely analytic function

$$f(\alpha) \equiv e^{|\alpha|^2/2}\langle\beta|\psi\rangle\langle\psi|\alpha\rangle = u(\alpha) + iv(\alpha). \tag{4}$$

Thus the functions u and v satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ with $\alpha \equiv x + iy$. Then we remind ourselves that the real part u of an analytic function fulfills Laplace's equation $u_{xx} + u_{yy} = 0$. Hence for u given on a closed boundary it is also known inside. Solving the just mentioned Cauchy-Riemann equations determines the imaginary part $v(\alpha)$ of the analytic function $f(\alpha)$, Eq. (4), inside the boundary. The constant of this integration is fixed by the requirement $f(\beta) \in \mathbb{R}$. Hence we have shown that the real-valued and operationally defined u function, Eq. (3), is equivalent to the knowledge of the complex coherent amplitude $\langle\beta|\psi\rangle \times \langle\psi|\alpha\rangle$ in a finite region of α space. Because of the overcompleteness of coherent states this is eventually sufficient [13,16] to determine $|\langle\beta|\psi\rangle|^2|\psi\rangle\langle\psi|$ which finally only needs to be normalized [17].

So far we have given a general prove that the u function, given as the interference term of a joint (X, P) measurement is a complete representation of the state $|\psi\rangle\langle\psi|$. However, an explicit relation between measurable values and the state was missing. In fact it sounds rather complicated to proceed via solutions of Laplace's equation and the Cauchy-Riemann relations. We will show now that the required integrations are quite simple when the u function, Eq. (3), is given on a circle $\alpha = Re^{i\phi}$ with fixed radius $R > 0$. The corresponding analytic function $f(\alpha)$, Eq. (4), for $|\alpha| < R$ then follows immediately from the Schwarz relation [18,19]

$$f(\alpha) = \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}u(Re^{i\phi})\frac{Re^{i\phi} + \alpha}{Re^{i\phi} - \alpha} + iv(\alpha = 0), \tag{5}$$

in which the constant $v(0)$ is again determined by the requirement $f(\beta) \in \mathbb{R}$. In order to simplify the following equations we now choose $\beta = 0$. This means that we define the u function, Eq. (3), relative to the vacuum overlap $\langle 0|\psi\rangle$ [20]. Consequently, we obtain $v(0) = 0$ in Eq. (5). Then we rewrite Eq. (5) in the form

$$f(\alpha) = \int_{-\pi}^{\pi}\frac{d\phi}{\pi}u(Re^{i\phi})\frac{1}{1 - \frac{\alpha}{Re^{i\phi}}} - \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}u(Re^{i\phi}). \tag{6}$$

Using the geometric series for $[1 - (\alpha/Re^{i\phi})]^{-1}$ yields

$$f(\alpha) = \sum_{n=0}^{\infty} f_n\frac{\alpha^n}{\sqrt{n!}} \tag{7}$$

with coefficients

$$f_n \equiv (2 - \delta_{n,0})\frac{\sqrt{n!}}{R^n}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}u(Re^{i\phi})e^{-in\phi}. \tag{8}$$

We emphasize that these coefficients are basically the Fourier components of the 2π periodic interference term $u(Re^{i\phi})$. Hence we have an explicit solution for the analytic function $f(\alpha)$, Eq. (4), which due to the corresponding power series expansion, Eq. (7), can be analytically continued on α space. Therefore, recalling the construction of $f(\alpha)$, Eq. (3), in terms of the coherent-state $|\alpha\rangle$ we also identify the unnormalized Fock coefficients

$$\langle 0|\psi\rangle\langle\psi|n\rangle = f_n \tag{9}$$

of our state $|\psi\rangle\langle\psi|$ in terms of the Fourier components f_n, Eq. (8). Hence we arrive at an explicit Fourier relation between the Fock representation and the 2π-periodic u function, Eq. (3). In this respect one can argue that it also provides a complete phase representation for the state $|\psi\rangle\langle\psi|$.

In Fig. 1 we show as an example the u function of a specific coherent-state superposition. The two main contributions are clearly visible. However, it would be desirable to extend the definition of the u function in such a way that we can apply it to distinguish such coherent superpositions from incoherent mixtures. Hence we need a generalization of the u function for mixed states $\hat{\rho}$. This again can be done operationally by choosing the quantum

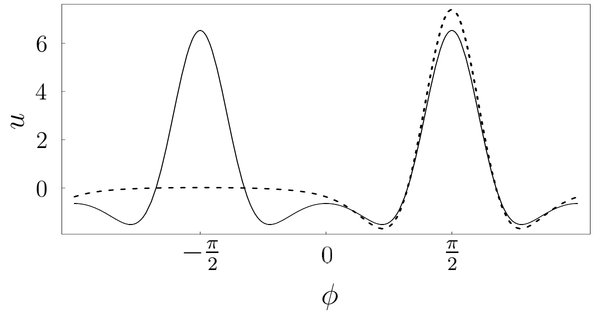

FIG. 1. The u function, Eq. (2), of the superposition state $\mathcal{N}(|e^{i\pi/2}\rangle + |e^{-i\pi/2}\rangle)$ (solid line) on a circle with radius $R = 3$ and with $\beta = 0$. We compare it to the corresponding function of an ordinary coherent-state $|e^{i\pi/2}\rangle$ (dashed line). The two main contributions of the superposition at $\phi = \pm\pi/2$ are clearly visible. We emphasize that these real-valued and periodic curves completely represent the underlying pure states.

ruler to be in state $|\varphi\rangle = \mathcal{N}(|\alpha\rangle + e^{i\gamma}|\alpha'^*\rangle)$, in which both coherent amplitudes α' and α are varied on closed curves and $\mathcal{N}$ denotes the respective normalization constant. The corresponding probability, Eq. (1), to find $X = 0$ and $P = 0$ in the state $\hat{\rho} \otimes |\varphi\rangle\langle\varphi|$ now leads to the interference term $\text{Re}\{\langle\alpha^*|\hat{\rho}|\alpha'\rangle\}$ which allows us to define

$$u(\alpha, \alpha') \equiv e^{|\alpha|^2/2+|\alpha'|^2/2}\text{Re}\{\langle\alpha^*|\hat{\rho}|\alpha'\rangle\}, \tag{10}$$

which is the real part of an analytic function $f(\alpha, \alpha')$. Consequently, this generalized u function again contains the complete state $\hat{\rho}$ if α and α' are given on closed curves. In particular, for two circles $\alpha = Re^{i\phi}$ and $\alpha' = R'e^{i\phi'}$ one arrives at the Schwarz relation

$$f(a, a') = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\frac{d\phi'}{2\pi}u(Re^{i\phi}, R'e^{i\phi}) \times \frac{(Re^{i\phi} + \alpha)(R'e^{i\phi'} + \alpha') - 2\alpha\alpha'}{(Re^{i\phi} - \alpha)(R'e^{i\phi'} - \alpha')}. \tag{11}$$

In analogy to the pure state case we then find the Fourier decomposition

$$\rho_{mn} = (2 - \delta_{m,0}\delta_{n,0})\frac{\sqrt{m!n!}}{R^m R'^n}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\frac{d\phi'}{2\pi} \times u(Re^{i\phi}, R'e^{i\phi'})e^{-im\phi-in\phi'} \tag{12}$$

for the Fock coefficients $\rho_{mn} \equiv \langle m|\hat{\rho}|n\rangle$. Note that the pure state case follows from Eq. (12) by putting $m = 0$. Hence a mixed state $\hat{\rho}$ can be completely represented by a real-valued quantity that depends on two periodic phase parameters. This also allows us to demonstrate the difference in the u function, Eq. (10), of a coherent and incoherent mixture, as shown in Fig. 2.

The additional contributions in the u function of the coherent superposition are clearly visible. They stem from the cross terms in the corresponding density operator and therefore demonstrate the coherence.

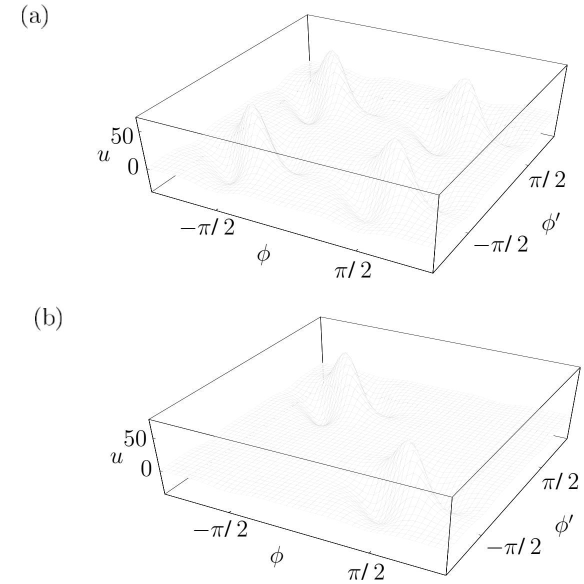

FIG. 2. Differences in the u functions of coherent and incoherent mixtures. In (a) we show the u function, Eq. (9), of the superposition state $\mathcal{N}(|e^{i\pi/2}\rangle + |e^{-i\pi/2}\rangle)$ with coherent states $|e^{\pm i\pi/2}\rangle$. One clearly sees the additional contributions as compared to (b) which depicts the u function of the incoherent mixture $\hat{\rho} = (|e^{i\pi/2}\rangle\langle e^{i\pi/2}| + |e^{-i\pi/2}\rangle\langle e^{-i\pi/2}|)/2$. Both plots are parametrized by $R = R' = 3$.

So far we have discussed the u function as a representation of quantum states. This naturally leads to the question how quantum dynamics looks like in this picture. Even though we cannot discuss the full theory here we shall outline the essential points. The von-Neumann equation $i\hbar\dot{\hat{\rho}} = [\hat{H}, \hat{\rho}]$ can in fact be rewritten in terms of the generalized u function, Eq. (10). This leads, however, for a general Hamiltonian $\hat{H}$ to a complicated system of equations which we shall analyze elsewhere [21]. Nevertheless, we can exemplify the dynamics for the standard system of a harmonic oscillator with $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$. In this case the evolution of a pure state is determined by the simple partial differential equation

$$\left(\frac{\partial}{\partial t} - \omega\frac{\partial}{\partial\phi}\right)u(Re^{i\phi}, t) = 0, \tag{13}$$

which only involves the ordinary u function, Eq. (3). This relation can be proven using the Fourier relation, Eq. (8). The corresponding solution

$$u(Re^{i\phi}, t) = u(Re^{i(\phi-\omega t)}, 0) \tag{14}$$

for the initial condition $u(Re^{i(\phi-\omega t)}, 0) = u(Re^{i\phi})$ demonstrates how clear the dynamics of a harmonic oscillator becomes in terms of this representation.

We finally note that the expansion of ruler states in terms of Fock states (harmonic oscillator eigenstates) is not essential for the definition of an operational u function. One could think of a ruler represented by any system with a complete set of eigenstates $|a_n\rangle$. A superposition of the form $\sum_n (\alpha^n/\sqrt{n!})|a_n\rangle$ would then be a suitable ruler since it leads to the same analyticity arguments that have been used in our analysis. Of course, the question remains how to prepare such states in general. We have chosen coherent states since for quantized electro-magnetic fields preparations are known [22,23]. Moreover (X, P) measurements can be performed at least in the optical domain using eight-port homodyning [24,25].

In conclusion, we have shown that it is possible to define an operational representation of quantum states guided by a conceptually simple position and momentum comparison between a quantum system and a quantum ruler. Hence we have closely followed the classical concept and found an alternative to real-valued quasi-probability distributions. The representation is based on real-valued quantum interference terms which dominate the quantum comparison for appropriate ruler states. Finally, we have briefly outlined the possibility to formulate quantum dynamics with the help of this representation. As the next step it is essential to further investigate the dynamics of specific quantum systems which might become particularly simple using the u function.

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