Anomalous Critical Exponents in the Anisotropic Ashkin-Teller Model

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We perform a rigorous computation of the specific heat of the Ashkin-Teller model in the case of small interaction and we explain how the universality-nonuniversality crossover is realized when the isotropic limit is reached. We prove that, even in the region where universality for the specific heat holds, anomalous critical exponents appear: for instance, we predict the existence of a previously unknown anomalous exponent, continuously varying with the strength of the interaction, describing how the difference between the critical temperatures rescales with the anisotropy parameter.

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More than half a century ago Ashkin and Teller (AT) [1] introduced their model as a generalization of the Ising model to a four component system. It describes a bidimensional lattice, each site of which is occupied by one of four kinds of atoms: A, B, C, D. Two neighboring atoms interact with an energy: ε_0 for AA, BB, CC, DD; ε_1 for AB, CD; ε_2 for AC, BD; and ε_3 for AD, BC. Fan [2] has shown that the AT model can be also written in terms of Ising variables $\sigma_{\mathbf{x}}^{(1)} = \pm 1$, $\sigma_{\mathbf{x}}^{(2)} = \pm 1$ located at each site of the lattice; its Hamiltonian can be written, if $\mathbf{x}, \mathbf{y}$ are nearest neighbor sites as

$$H^{AT} = -\sum_{\mathbf{x},\mathbf{y}} J^{(1)}\sigma_{\mathbf{x}}^{(1)}\sigma_{\mathbf{y}}^{(1)} + J^{(2)}\sigma_{\mathbf{x}}^{(2)}\sigma_{\mathbf{y}}^{(2)} + \lambda\sigma_{\mathbf{x}}^{(1)}\sigma_{\mathbf{y}}^{(1)}\sigma_{\mathbf{x}}^{(2)}\sigma_{\mathbf{y}}^{(2)}, \qquad (1)$$

with $J^{(1)} = \beta(\varepsilon_0 + \varepsilon_1 - \varepsilon_2 - \varepsilon_3)/4$, $J^{(2)} = \beta(\varepsilon_0 + \varepsilon_2 - \varepsilon_1 - \varepsilon_3)/4$, $\lambda = \beta(\varepsilon_0 + \varepsilon_3 - \varepsilon_1 - \varepsilon_2)/4$, and β is the inverse temperature. The AT model is then equivalent to two Ising models coupled by an interaction quartic in the spins; the case in which the two Ising subsystems are identical $J^{(1)} = J^{(2)}$ is called *isotropic*, the opposite case *anisotropic*. When the coupling λ is equal to 0, the AT reduces to two independent Ising models and it has of course *two* critical temperatures if $J^{(1)} \neq J^{(2)}$. We shall consider the case of $J^{(1)}, J^{(2)}$ fixed and positive (for definiteness).

Layers of atoms and molecules adsorbed on clean surfaces, like submonolayers of Se adsorbed on Ni, are believed to constitute physical realizations of the AT model [3–5]; theoretical results on it can explain the phase diagrams of such systems, experimentally obtained by means of electron diffraction techniques. As for the Ising model, the importance of AT is also in providing a conceptual laboratory in which the highly nontrivial phenomenon of phase transitions can be understood quantitatively in a relatively manageable model; in particular, it has attracted great theoretical interest because is a simple and nontrivial generalization of the Ising and four-state Potts models.

Contrary to many 2D models in statistical mechanics like the Ising, the 6, or the 8 vertex models [6], in which remarkable exact solutions give us very detailed information about the behavior of thermodynamical functions, there are no exact results on the AT model except for the trivial $\lambda = 0$ case. It is believed [7] that the AT has *two* critical temperatures for $J^{(1)} = J^{(2)}$ which coincide at the isotropic point $J^{(1)} = J^{(2)}$. Moreover, it was conjectured by Kadanoff [8] and Baxter [6] that the critical properties in the anisotropic and in the isotropic case are completely different; in the first case the critical behavior should be described in terms of *universal* critical indices (identical to those of the 2D Ising model), while in the isotropic case the critical behavior should be *nonuniversal* and described in terms of indexes which are nontrivial functions of λ. In other words, the AT model should exhibit a *universal-nonuniversal* crossover when the isotropic point is reached.

Evidence for the validity of nonuniversal behavior in the isotropic case was given in [9] (using second order renormalization group arguments) and in [10,11] (by a heuristic mapping into the massive Luttinger model describing one dimensional interacting fermions in the continuum). The anisotropic case was studied numerically by Migdal-Kadanoff renormalization group [5], Monte Carlo renormalization group [12], finite size scaling [13]; such results give evidence of the fact that, far away from the isotropic point, AT has two critical points and belongs to the same universality class of the Ising model but give essentially no information on the critical behavior when the anisotropy is small.

In this Letter, we present a *rigorous* derivation of the specific heat for the AT model, valid for small interaction λ and any anisotropy. We find indeed that in the anisotropic case the specific heat is singular in the correspondence of two critical temperatures, and the divergence is *logarithmic* as in the Ising model, in agreement with universality hypothesis. Nevertheless, even in the region where universality holds, *anomalous* critical exponents appear; for instance the difference between the two critical temperatures rescales with the anisotropy parameter with a nonuniversal critical exponent. The presence of such critical exponents also in the universality region clarify how the universal-nonuniversal crossover is realized when the isotropic limit is reached.

Such results are found by the new methods introduced in [14,15] to study 2D statistical mechanics models which can be considered perturbations of the Ising model. These methods take advantage from the fact that such systems can be exactly mapped in models of weakly interacting relativistic fermions in $d = 1 + 1$ on a lattice. The mapping was known since long time (see Refs. [10,16–18]); however, in recent years great progress has been achieved in the evaluation of Grassmann integrals of interacting models, in the context of quantum field theory and solid state physics (see Refs. [19–21]), and one can take advantage of this new technology to get information about 2D statistical mechanics models. This provides the only method to get rigorous quantitative information on the critical properties of such systems if an exact solution is lacking, as in the present case. The algorithm is based on multiscale analysis and allows us to prove *convergence* of the expansion for the energy-energy correlation functions and for the specific heat *up to the critical temperature*; essential ingredients of our analysis are cancellations due to anticommutativity of fermionic variables and approximate *Ward identities* [22], guaranteeing that the flow of the effective coupling constants is not diverging in the infrared region. We stress that our method applies to a large class of perturbations of the 2D Ising model, and for sake of definiteness we restrict our analysis to AT.

In order to present our result, we find it convenient to introduce the variables

$$t = \frac{t^{(1)} + t^{(2)}}{2}, \qquad u = \frac{t^{(1)} - t^{(2)}}{2}, \tag{2}$$

with $t^{(j)} = \tanh J^{(j)}$, $j = 1, 2$. The parameter t has the role of a *reduced temperature* and u measures the *anisotropy* of the system. We shall consider the free energy or the specific heat as functions of t, u, λ. When $\lambda = 0$ the specific heat C_v can be immediately computed from the Ising model exact solution; C_v is diverging at $t = t_c^{\pm} = \sqrt{2} - 1 \pm |u|$ and near the critical temperatures the specific heat shows a logarithmic divergence: $C_v \simeq -C \log|t - t_c^{\pm}|$, where $C > 0$. If the anisotropy is strong the two Ising subsystems have very different critical temperatures, hence one can expect that if one system is almost critical the second one will be out of criticality; then mean field arguments based on the fact that two Ising are coupled by a density-density interaction suggest that the effect of the coupling is to change at most the value of the critical temperatures. On the other hand if the anisotropy is small the two system will become critical almost at the same temperature and the properties of the system could change drastically.

Our main result is the following theorem; the detailed proof can be found in [15,23].

Theorem: *For λ small enough the AT model admits two critical points of the form*

$$t_c^{\pm}(\lambda, u) = \sqrt{2} - 1 + \nu(\lambda) \pm |u|^{1+\eta}[1 + \delta(\lambda, u)].$$

Here ν and δ are $O(\lambda)$ corrections and $\eta = -b\lambda + O(\lambda^2)$ with $b > 0$. If $t \neq t_c^{\pm}$ the free energy of the model is analytic in λ, t, u, and the specific heat C_v is equal to

$$-F_1 \Delta^{2\eta_c} \log \frac{|t - t_c^-| \cdot |t - t_c^+|}{\Delta^2} + F_2 \frac{1 - \Delta^{2\eta_c}}{\eta_c} + F_3, \tag{3}$$

where $2\Delta^2 = (t - t_c^-)^2 + (t - t_c^+)^2$; $\eta_c = a\lambda + O(\lambda^2)$, $a \neq 0$; and F_1, F_2, F_3 are functions of t, u, λ, bounded above and below by $O(1)$ constants.

(1) First note that the location of the critical points is dramatically changed by the interaction. The difference of the interacting critical temperatures normalized with the free one $G(\lambda, u) \equiv [t_c^+(\lambda, u) - t_c^-(\lambda, u)]/[t_c^+(0, u) - t_c^-(0, u)]$ rescales with the anisotropy parameter as a power law $\sim |u|^{\eta}$, and in the limit $u \to 0$ it vanishes or diverges, depending on the sign of λ [this is because $\eta = -b\lambda + O(\lambda^2)$, with $b > 0$]. In Fig. 1 we plot the qualitative behavior of $G(\lambda, u)$ as a function of u, for two different values of λ (i.e., we plot the function u^{η}, with $\eta = 0.3, -0.3$, respectively).

As far as we know, the existence of the critical index $\eta(\lambda)$ was not known in the literature, even at a heuristic level.

(2) There is universality for the specific heat, in the sense that it diverges logarithmically at the critical points, as in the Ising model. However, the coefficient of the log is *anomalous*: in fact if t is near to one of the critical temperatures $\Delta \simeq \sqrt{2}|u|^{1+\eta}$ so that the coefficient in front of the logarithm behaves like $\sim |u|^{2(1+\eta)\eta_c}$, with η_c a new anomalous exponent $O(\lambda)$; in particular, it is

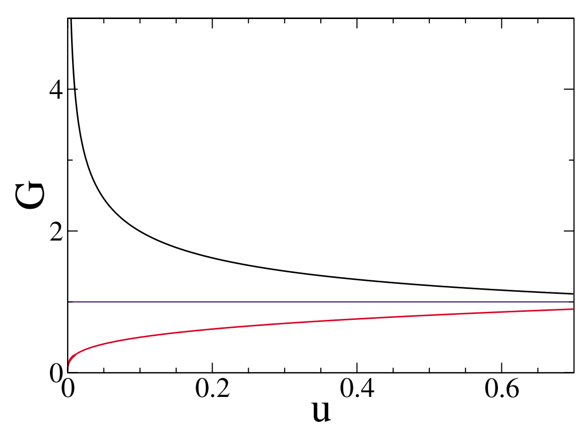

FIG. 1 (color online). The behavior of the difference G between the interacting critical temperatures normalized to the free one, for two different values of λ; depending on the sign of the interaction, it diverges or vanishes in the isotropic limit.

vanishing or diverging as $u \to 0$ depending on the sign of λ. We can say that the system shows an *anomalous universality* which is a sort of new paradigmatic behavior: the singularity at the critical points is described in terms of universal critical indexes; nevertheless, in the isotropic limit $u \to 0$, some quantities, such as the difference of the critical temperatures and the constant in front of the logarithm in the specific heat, scale with anomalous critical indexes, and they vanish or diverge, depending on the sign of λ.

(3) Equation (3) clarifies how the universality-nonuniversality crossover is realized as $u \to 0$. When $u \neq 0$ only the first term in Eq. (3) can be log singular in correspondence of the two critical points; however, the logarithmic term dominates on the second one only if t varies inside an extremely small region $O(|u|^{1+\eta}e^{-c/|\lambda|})$ around the critical points [here c is a positive $O(1)$ constant]. Outside such region the power law behavior corresponding to the second addend dominates. When $u \to 0$ one recovers the power law decay found in the isotropic case

$$C_v \simeq F_2 \frac{1 - |t - t_c|^{2\eta_c}}{\eta_c}.$$

In Fig. 2 we plot the qualitative behavior of C_v as a function of t. The three graphs are plots of Eq. (3), with $F_1 = F_2 = 1$, $F_3 = 0$, $u = 0.01$, $t_c^{\pm} = \sqrt{2} - 1 \pm |u|^{1+}$ and $\eta = \eta_c = 0.1, 0, -0.1$, respectively; the central curve corresponds to the case $\eta = 0$, the upper one to $\eta < 0$ and the lower to $\eta > 0$.

We now sketch the proof of the above theorem (for a detailed proof we refer to [15,23]). We start from the well-known representation of the Ising model free energy in terms of a sum of *Pfaffians* [24] which can be equivalently written (see Refs. [17,18]) as *Grassmann functional integrals*, formally describing massive noninteracting Majorana fermions $\psi, \overline{\psi}$ on a lattice with action

$$\sum_{\mathbf{x}} \frac{t}{4}[\psi_{\mathbf{x}}(\partial_1 - i\partial_0)\psi_{\mathbf{x}} + \overline{\psi}_{\mathbf{x}}(\partial_1 + i\partial_0)\overline{\psi}_{\mathbf{x}} - 2i\overline{\psi}_{\mathbf{x}}(\partial_1 + \partial_0)\psi_{\mathbf{x}}] + i(\sqrt{2} - 1 - t)\overline{\psi}_{\mathbf{x}}\psi_{\mathbf{x}}, \qquad (4)$$

where ∂_j are discrete derivatives; criticality corresponds to the massless case. If $\lambda = 0$ the free energy and specific heat of the AT model can be written as sum of Grassmann integrals describing *two* kinds of Majorana fields, with masses $m^{(1)} = t^{(1)} - \sqrt{2} + 1$ and $m^{(2)} = t^{(2)} - \sqrt{2} + 1$.

If $\lambda \neq 0$ again the free energy and the specific heat can be written as Grassmann integrals, but the Majorana fields are *interacting* with a short range potential. By performing a suitable change of variables and integrating out the ultraviolet degrees of freedom, the effective action can be written as

$$Z_1 \sum_{\mathbf{x},\omega,\alpha} [\psi^+_{\omega,\mathbf{x}}(\partial_1 - i\omega\partial_0)\psi^-_{\omega,\mathbf{x}} - i\omega\sigma_1\psi^+_{\omega,\mathbf{x}}\psi^-_{-\omega,\mathbf{x}} + i\omega\mu_1\psi^\alpha_{\omega,\mathbf{x}}\psi^\alpha_{-\omega,-\mathbf{x}} + \lambda_1\psi^+_{1,\mathbf{x}}\psi^-_{1,\mathbf{x}}\psi^+_{-1,\mathbf{x}}\psi^-_{-1,\mathbf{x}}] + \mathcal{W}_1,$$

where $\alpha = \pm$ is a *creation-annihilation* index and $\omega = \pm 1$ is a *quasiparticle* index. σ_1 and μ_1 have the role of two *masses* and it holds $\sigma_1 = O(t - \sqrt{2} + 1) + O(\lambda)$, $\mu_1 = O(u)$. $\mathcal{W}_1$ is a sum of monomials of ψ of arbitrary order, with kernels which are *analytic functions* of λ_1; analyticity is a very nontrivial property obtained exploiting anticommutativity properties of Grassmann variables via *Gram inequality* for determinants. The $\psi^{\pm}$ are *Dirac* fields, which are combinations of the Majorana variables $\psi^{(j)}, \overline{\psi}^{(j)}$, $j = 1, 2$, associated with the two Ising subsystems.

One can compute the partition function by expanding the exponential of the action in Taylor series in λ and naively integrating term by term the Grassmann monomials, using the Wick rule; however, such a procedure gives poor bounds for the coefficients of this series that, in the thermodynamic limit, can converge only far from the critical points.

In order to study the critical behavior of the system we perform a multiscale analysis involving nontrivial resummations of the perturbative series. The first step is to decompose the propagator $\hat{g}(\mathbf{k})$ as a sum of propagators more and more singular in the infrared region, labeled by an integer $h \leq 1$, so that $\hat{g}(\mathbf{k}) = \sum_{h=-\infty}^{1} \hat{g}^{(h)}(\mathbf{k})$, $\hat{g}^{(h)}(\mathbf{k}) \sim \gamma^{-h}$. We compute the Grassmann integrals defining the partition function by iteratively integrating the propagators $\hat{g}^{(1)}, \hat{g}^{(0)}, \ldots$. After each integration step we rewrite the partition function in a way similar to the last equation, with $Z_h, \sigma_h, \mu_h, \lambda_h, \mathcal{W}_h$ replacing $Z_1, \sigma_1, \mu_1, \lambda_1, \mathcal{W}_1$, in particular, the masses and the wave function renormalization are modified; the structure of the action is preserved because of symmetry properties; moreover $\mathcal{W}_h$ is shown to be a sum of monomials of ψ of

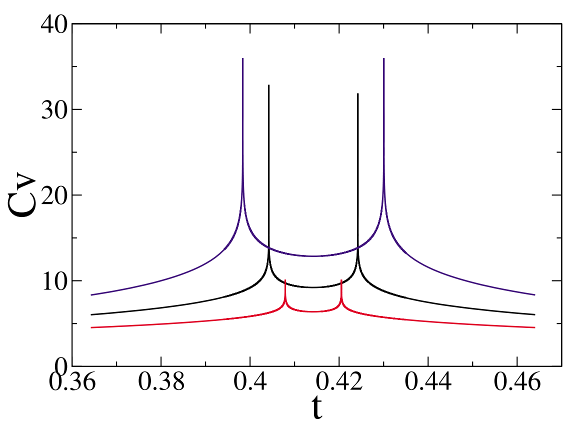

FIG. 2 (color online). The behavior of the specific heat C_v for three different values of λ, showing the log singularities at the critical points. In the isotropic limit the two critical points tend to coincide; the lower curve becomes continuous while the upper develops a power law divergence.

arbitrary order, with kernels decaying in real space on scale γ^{-h}, which are *analytic functions* of $\{\lambda_h, \ldots, \lambda_1\}$, if λ_k are small enough, $k \geq h$, and $|\sigma_k|\gamma^{-k}, |\mu_k|\gamma^{-k} \leq 1$; again analyticity follows from Gram-Hadamard type of bounds.

All of the above construction is based on the crucial property that the effective interaction at each scale does not increase $|\lambda_h| \leq 2|\lambda|$; such a property is a consequence of the validity of some nonperturbative *approximate Ward identities* [22]; "approximate" refers to the fact that, because of the presence of masses and of an ultraviolet cutoff, the Ward identities are different from the usual formal ones; the error terms are shown to be small, in a suitable sense. For σ_h, μ_h, Z_h, we find that, under the iterations, they evolve as $\sigma_h \simeq \sigma_1 \gamma^{b_2 \lambda h}$, $\mu_h \simeq \mu_1 \gamma^{-b_2 \lambda h}$, $Z_h \simeq \gamma^{-b_1 \lambda^2 h}$, with b_1, b_2 explicitly computable in terms of a convergent power series.

We perform the iterative integration described above up to a scale h_1^* such that $(|\sigma_{h_1^*}| + |\mu_{h_1^*}|)\gamma^{-h_1^*} = O(1)$. For scales lower than h_1^* we return to the description in terms of the original Majorana fermions $\psi^{(1,\leq h_1^*)}$, $\psi^{(2,\leq h_1^*)}$ associated with the two Ising subsystems. One of the two fields [say $\psi^{(1,\leq h_1^*)}$] is massive on scale h_1^* (so that the Ising subsystem with $j = 1$ is "far from criticality" on the same scale); then we can integrate the massive Majorana field $\psi^{(1,\leq h_1^*)}$ without any further multiscale analysis, obtaining an effective theory of a single Majorana field with mass $|\sigma_{h_1^*}| - |\mu_{h_1^*}|$, which can be arbitrarily small; this is equivalent to saying that on scale h_1^* we have an effective description of the system as a single perturbed Ising model with *anomalous* parameters near criticality. The integration of the scales $\leq h_1^*$ is performed again by a multiscale decomposition similar to the one just described; an important feature is, however, that there are no more quartic marginal terms, because the anticommutativity of Grassmann variables forbids local quartic monomials of a single Majorana fermion. Criticality is found when the effective mass on scale $-\infty$ is vanishing; the values of t, u for which this happens are found by solving a nontrivial implicit function problem.

Technically it is an interesting feature of this problem that there are two regimes in which the system must be described in terms of different fields: a first one in which the natural variables are Dirac-Grassmann variables, and a second one in which they are Majorana; the scale h_1^* separating the two regimes is dynamically generated by the iterations. In the first regime the two entangled Ising subsystems are undistinguishable, the natural description is in terms of Dirac variables, and the effective interaction is marginal. In the integration of such scales nonuniversal indexes appear. In the second region the two Ising subsystems really look different, one appears to be (almost) at criticality and the other far from criticality on the same scale. The parameters of the two subsystems are deeply changed (in an anomalous way) by the previous integration. In this region the effective interaction is irrelevant.

In conclusion, we have presented some new rigorous results on the critical behavior in the Ashkin-Teller model, for weak coupling and any value of the anisotropy. Via multiscale integration methods we have computed the specific heat and the location of the critical temperatures in terms of *convergent* power series and we have predicted the existence of an unknown critical exponent describing the scaling of the gap between the critical temperatures in the isotropic limit. Moreover, we gave a detailed description of the crossover between the universal critical behavior holding in the anisotropic case and the anomalous nonuniversal behavior holding in the isotropic limit.

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