Superscaling of Percolation on Rectangular Domains

Hiroshi Watanabe,[1,2] Satoshi Yukawa,[2] Nobuyasu Ito,[2] and Chin-Kun Hu[3,4,*]

[1]Department of Complex Systems Science, Graduate School of Information Science, Nagoya University, Furouchou, Chikusa-ku, Nagoya 464-8601, Japan

[2]Department of Applied Physics, School of Engineering, The University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-8656, Japan

[3]Institute of Physics, Academia Sinica, Nankang, Taipei 11529, Taiwan

[4]National Center for Theoretical Sciences at Taipei, Physics Division, National Taiwan University, Taipei 10617, Taiwan

(Received 21 June 2004; published 2 November 2004)

For percolation on $(RL) \times L$ two-dimensional rectangular domains with a width L and aspect ratio R, we propose that the existence probability of the percolating cluster $E_p(L, \varepsilon, R)$ as a function of L, R, and deviation from the critical point ε can be expressed as $F(\varepsilon L^{y_t} R^a)$, where $y_t \equiv 1/\nu$ is the thermal scaling power, a is a new exponent, and F is a scaling function. We use Monte Carlo simulation of bond percolation on square lattices to test our proposal and find that it is well satisfied with $a = 0.14(1)$ for $R > 2$. We also propose superscaling for other critical quantities.

DOI: 10.1103/PhysRevLett.93.190601 PACS numbers: 05.50.+q, 24.10.Lx, 89.75.Da

Percolation models are related to many interesting problems in sciences and are ideal systems for studying universality and scaling in critical phenomena [1–6]. For example, it has been found that the excess number of clusters over the bulk value n_c is a universal quantity [3]; it has also been found that quite different percolation models [4,5] have universal finite-size scaling functions [7] for their existence probability of the percolating clusters as far as the domains of the percolation models have the same boundary conditions and aspect ratio. Here we will study superscaling of percolation on domains with different aspect ratios. For percolation on a $(RL) \times L$ two-dimensional rectangular domains with horizontal length (RL), vertical length L, and aspect ratio R, we propose that the existence probability $E_p(L, \varepsilon, R)$ as a function of L, R, and deviation from the critical point ε can be expressed as $F(\varepsilon L^{y_t} R^a)$, where $y_t \equiv 1/\nu$ is the thermal scaling power, a is a new exponent, and F is a scaling function. We use the histogram Monte Carlo (MC) simulation method [8] to study bond percolation on square (SQ) lattices to test our proposal and find that it is well satisfied with $a = 0.14(1)$ for $R > 2$. We also find that the percolation probability P for the same model has nice superscaling behavior with a new exponent $b = 0.05(1)$.

Consider a two-dimensional percolation problem on a $L_1 \times L_2$ domain, where $L_1 = RL$ and $L_2 = L$. The correlation function $C(r)$ is the probability that two sites at a distance r are in the same cluster. This $C(r)$ is well approximated to be

$$C(r) = A \exp(-r/\xi), \tag{1}$$

with a constant A [9] and a correlation length ξ. Near the critical point, ξ is written as

$$\xi = B|\rho - \rho_c|^{-\nu}, \tag{2}$$

with a nonuniversal constant B. When the system does not percolate, any pair of sites at the top and bottom of the system are not in the same cluster. Ignoring multipoint correlations [10], we can write the probability that there is no percolating clusters, $1 - E_p$, as

$$\begin{aligned} 1 - E_p &= \prod_{x=0}^{L_1} \prod_{y=0}^{L_1} [1 - C(\sqrt{L_2^2 + (x-y)^2})] \\ &= \prod_{x=0}^{L_1} \prod_{y=0}^{L_1} \left[1 - A \exp\left(-\frac{L_2}{\xi} \sqrt{1 + \frac{(x-y)^2}{L_2^2}} \right) \right]. \end{aligned} \tag{3}$$

In the continuous limit, we have:

$$\log(1 - E_p) \propto \int_0^{L_1} dx \int_0^{L_1} dy \log\left[1 - A \exp\left(-\frac{L_2}{\xi} \times \sqrt{1 + \frac{(x-y)^2}{L_2^2}} \right) \right].$$

With substitution $u = (x + y)/L_2$ and $v = (x - y)/L_2$ and the saddle-point approximation near $v = 0$, we have

$$\log(1 - E_p) \propto - \int_{-(L_1/L_2)^\omega}^{(L_1/L_2)^\omega} dv \exp\left(-\frac{L_2}{2\xi} v^2 \right). \tag{4}$$

Here, the range where the value of $\exp(-L_2 v^2/2\xi)$ contributes to the integration (4) is assumed to have power law dependence on the aspect ratio. Therefore, the integration range of v extends from $v = -(L_1/L_2)^\omega$ to $(L_1/L_2)^\omega$, with an exponent ω (see Fig. 1). With the aspect ratio $R = L_1/L_2$ and $L \equiv L_2$, Eq. (4) becomes

$$\log(1 - E_p) \propto - \int_{-R^\omega}^{R^\omega} dv \exp\left(-\frac{L}{2\xi} v^2 \right) \tag{5}$$

$$\sim -\mathrm{erf}\left(\sqrt{\frac{L}{2\xi}} R^\omega \right). \tag{6}$$

With $e^{-t} \simeq 1 - t$ for $t \ll 1$, E_p finally becomes

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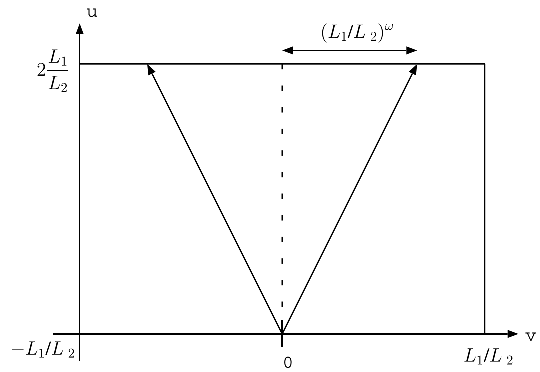

FIG. 1. Schematic drawing for the meaning of ω. The range of v is taken from $-(L_1/L_2)^\omega$ to $(L_1/L_2)^\omega$ where the value of $\exp[-L_2 v^2/(2/\xi)]$ contributes to the integration in Eq. (4).

$$E_p \sim \mathrm{erf}\left(\sqrt{\frac{\varepsilon^\nu L R^{2\omega}}{2B}}\right). \tag{7}$$

Equation (7) implies that E_p can be written as

$$E_p(L, \varepsilon, R) \sim F(\varepsilon L^{y_t} R^a), \tag{8}$$

with $a = 2\omega/\nu$.

To test the previous argument, we use the histogram MC simulation [8] to calculate E_p for bond percolation model on $LR \times L$ SQ lattices with system sizes $L = 64$, 128, and 256, and $R = 1$, 2, 4, 8, and 16. Free boundary conditions are taken for the vertical direction, and both free and periodic boundary conditions are studied for the horizontal direction. Averages are taken over 100 000 MC samples at each density. The results are shown in Fig. 2,

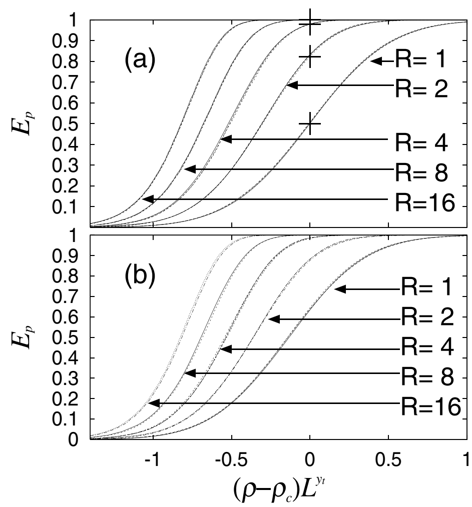

FIG. 2. Existence probability E_p of the system with (a) free and (b) periodic boundary conditions. For each R, three system sizes $L = 64\,128$ and 256 are shown as solid, dotted, and dashed line, respectively. Cardy's exact results [11] are also shown as + for comparing with our results.

which are consistent with Cardy's exact critical E_p for free boundary conditions in both directions [11].

In the following, instead of the critical point for the thermodynamic bulk system ρ_c, the effective critical density ρ_c' defined by $E_p(\rho_c') = 0.5$ for free and $E_p(\rho_c') = 0.636\,65$ for periodic boundary cases [12], is used in order to correct for finite-size effects. This ρ_c' is the density when ξ becomes the order of the system size L. Thus the ρ_c' has L dependence like $\rho_c' = \rho_c - dL^{-y_t}$ with a constant d, which can be written as:

$$(\rho - \rho_c')L^{y_t} = (\rho - \rho_c)L^{y_t} + d. \tag{9}$$

Thus the difference between using ρ_c and using ρ_c' is a translation in the scaling variable. While d does not depend on L and ρ, it is a function of R, see Fig. 3.

We show E_p as a function of $(\rho - \rho_c')L^{y_t}R^a$ in Fig. 4, where $y_t = 3/4$ [1] and the scaling power a is determined to be 0.14(1) for both free and the periodic boundary conditions. Figure 4 shows good scaling behavior when $R > 2$. Since our argument is rather general, it will be valid for other percolation models studied in Refs. [4,5].

The systems with small R do not have good scaling behavior, which can be understood from boundary effects of the systems. The behavior of the correlation function of Eq. (1) is assumed to be independent of the aspect ratio. However, this assumption is not valid near the boundary. The correlation functions $C(r)$ of the systems with different aspect ratios are shown in Fig. 5. While all curves show the equivalent behavior for small r, they show difference near boundaries. This difference is more remarkable in the cases of $R = 1$ and 2 compared to cases of larger R. Finally, the assumption is justified only when the systems have large R.

The scaling behaviors in Fig. 4 would be improved by taking into account higher-order corrections to the scaling. In Eq. (8), the scaling variable ρ' has a linear form as

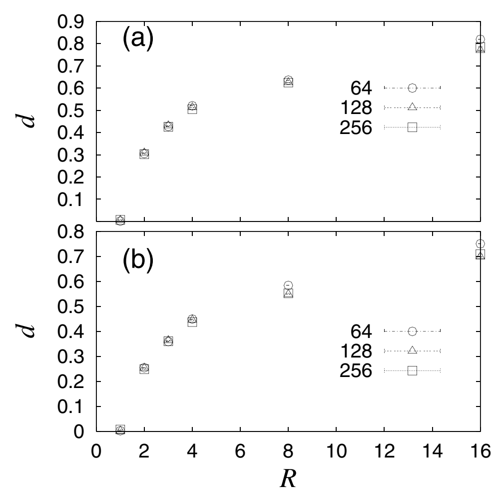

FIG. 3. Aspect ratio dependence of d in Eq. (9) for (a) free and (b) periodic boundary conditions.

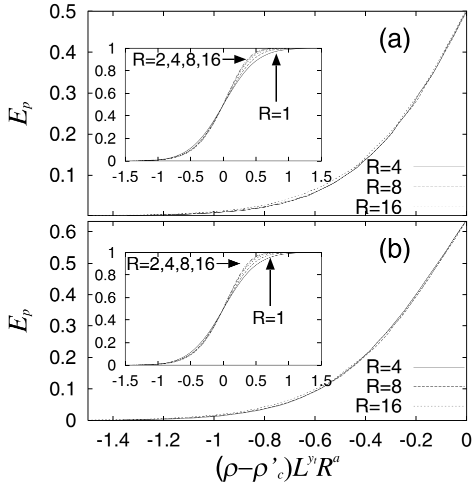

FIG. 4. Scaling plots of existence probabilities E_p with $y_t = 3/4$ and $a = 0.14(1)$ for $L = 256$ systems with (a) the free and (b) the periodic boundary conditions. All data (system sizes $L = 64$, 128 and 256, and aspect ratios $R = 1$, 2, 4, 8, and 16) are shown in the inset.

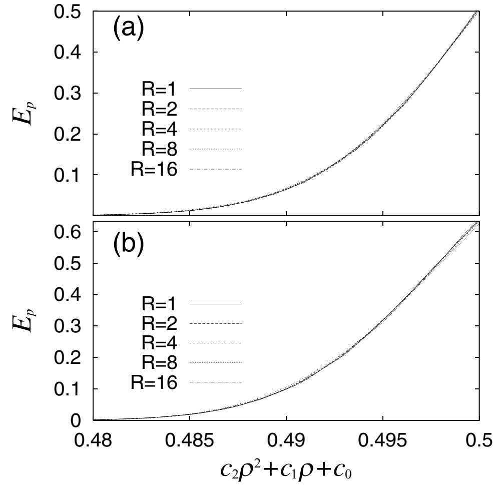

FIG. 6. Nonlinear scaling results of existence probability E_p of $L = 256$ system with (a) free and (b) periodic boundary conditions. The scaling form is $c_2\rho^2 + c_1\rho + c_0$. The coefficients c_i are listed in Table. I.

$\rho' = (\rho - \rho_c)L^{y_t}R^a \equiv c_0 + c_1\rho$. Now we consider higher-order for the scaling variables as $\rho' \equiv f(\rho) = c_0 + c_1\rho + c_2\rho^2$ [13]. Scaling results using such scaling variables are shown in Fig. 6. It shows pretty good scaling behavior, including systems with small R. This result implies that the scaling between different aspect ratios is not linear as proposed in Eq. (8). One of the reasons why the scaling is nonlinear is that the correlation function is not isotropic in rectangular domains. It is difficult to consider such aspect ratio dependence precisely; it would be one of the problems for further studies.

If a physical quantity Q of a thermodynamic system is of the form $Q(t) \sim t^x$ near the critical point $t = 0$, finite-size scaling theory states that for a finite system of linear dimension L, the corresponding quantity $Q(L, t)$ is of the form [1,7] $Q(L, t) \sim L^{-xy_t}F(tL^{y_t})$, where $F(y)$ $(y = tL^{y_t})$ is the scaling function. For E_p, $x = 0$ [1,4]. Based on general idea of finite-size scaling, we propose that Q of a $RL \times L$ critical system, $Q(L, R, t)$, can be written as

$$Q(L, R, t) \sim (L^{y_t}R^b)^{-x}F(tL^{y_t}R^b), \tag{10}$$

where b is a new exponent for Q. To test Eq. (10), we calculate the percolation probability P for bond percolation (here $x = \beta = 5/36$ [1]) on $RL \times L$ SQ lattices and present the data in Fig. 7, which shows that Eq. (10) is well satisfied with $b = 0.05(1)$.

It has been shown that the Ising-type spin models and lattice hard-core particle model can be considered as correlated percolation models [14], and some of such

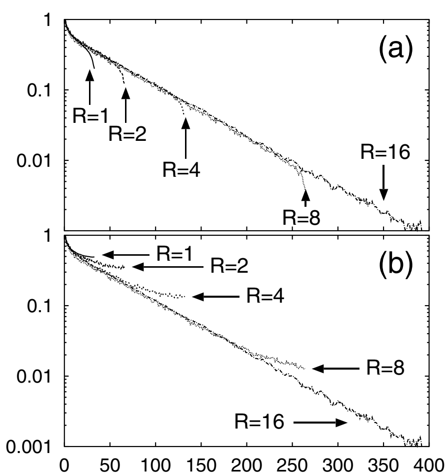

FIG. 5. The correlation function $C(r)$ with a distance r from a center point of the system for (a) free and (b) periodic boundary conditions. Decimal logarithm is taken for the vertical axis. System size L is 64 and aspect ratios R are 1, 2, 4, 8, and 16. Averages are taken over 100 000 MC samples.

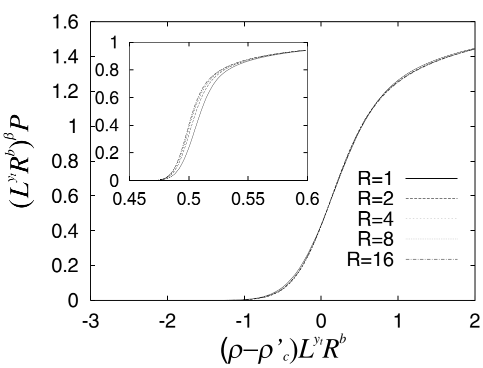

FIG. 7. Scaling plots of percolation probability P for $R = 1$, 2, 4, 8, and 16, and $L = 128$ with $y_t = 3/4$ and $b = 0.05(1)$. Data before scaling are shown in the inset. As in Fig. 4, ρ'_c (instead of ρ_c) is used to define the horizontal variable to reduce the finite-size effects.

TABLE I. List of coefficients of the nonlinear scaling with (a) free and (b) periodic boundary conditions. Coefficients of third order c_3 are also shown in order to show that the quadratic form is enough for the nonlinear scaling [13].

R	1	2	4	8	16
(a) free conditions					
c_0	0	1.4(1)	2.3(1)	3.9(1)	5.9(2)
c_1	1	−5.0(2)	−8.6(5)	−15.6(6)	−24(1)
c_2	0	6.4(5)	10.2(4)	17.6(6)	26(1)
c_3	0	−0.046(3)	0.033(4)	0.070(2)	0.100(6)
(b) periodic conditions					
c_0	0	1.66(5)	4.6(1)	7.74(4)	6.85(5)
c_1	1	−5.95(5)	−18.2(4)	−31.2(7)	−27.8(7)
c_2	0	7.25(7)	20.0(4)	33.5(7)	30.2(8)
c_3	0	0.003(5)	0.059(4)	0.130(4)	0.150(7)

correlated percolation models have been found to have good finite-size scaling behaviors [15]. It is of interest to study superscaling in such correlated percolation models and other critical systems.

We would like to thank C.-Y. Lin, W.-J. Ma, S. Miyashita, and Y. Okabe for fruitful discussions. The simulations were partially carried out on SGI 2800 at the Supercomputer Center, Institute for Solid State Physics, University of Tokyo. This work was partially supported by National Science Council in Taiwan under Grant No. NSC 92-2112-M 001-063, the Nestlé Science Promotion Committee, the Grant-in-Aid for Scientific Research (C), No. 15607003, of Japan Society for the Promotion of Science, and the Grant-in-Aid for Young Scientists (B), No. 14740229, of the Ministry of Education, Culture, Sports, Science, and Technology of Japan. H.W. would like to thank Physics Division of NCTS at Taipei for hospitality where part of this work was done.

*Corresponding author with electronic address: huck@phys.sinica.edu.tw.

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[9] In general, A is a polynomial function of r. However, in off critical region and with a large r, the approximation that A is constant is well justified (see Fig. 5).

[10] If site X is at the top boundary, and Y and Z are at the bottom boundary of the system, then the probability Pr that the system percolates via X to Y or X to Z is $Pr = C(X, Y) + C(X, Z) - C(X, Y, Z)$; multipoint correlations $C(X, Y, Z)$ are ignored in Eq. (4) as $Pr \sim C(X, Y) + C(X, Z)$, which is justified for large systems.

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[13] The coefficients c_i are determined by the Legendre Polynomial expansion. Weights of higher-order $(i > 2)$ are confirmed to be small enough comparing to those of lower order. It justifies that the mapping function $f(\rho)$ is well approximated by the quadratic function.

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