Quantum Phase Control of Entanglement

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A method of phase control of entanglement in two-qubit systems is proposed. We show that by changing a relative phase of the pulses that drive the transitions in a two-qubit system with closed-loop couplings, one can control entanglement at will. The method relies on adiabatic dynamics via time-delayed pulse sequences and can be implemented with both resonant and nonresonant transitions.

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One of the requirements of a quantum computer is the entanglement of qubits in a quantum register. This goal underneaths extensive efforts toward creating two particle entangled states in various composite quantum systems, potential candidates for quantum computation [1–3], as well as many-particle entangled states [4–10]. Another reason for growing interest in creating entangled states is that a subgroup of these states, the so-called spin-squeezed states, can lead to an improvement in high-precision measurements below the standard quantum limit (in an ideal situation, the Heisenberg limit can be achieved) [7].

In principle, entanglement is the most natural state in quantum systems: any two or more quantum systems interacting for some time become entangled. That is, the wave function of the combined system cannot be written as a direct product of wave functions of its constituents. Obviously it is crucial for quantum information processing to be able to create entangled states in a controllable way. The problem of controlling entanglement is directly connected to the problem of coherent control of population transfer in multilevel systems. To understand this one-to-one correspondence, one may consider a system of two interacting $1/2$ spins in a static magnetic field (two qubits). The energy levels of the combined system form a four-level system. As a result, the problem of controlling entanglement is mapped onto the problem of creating specific coherent superpositions in the four-level system.

In general, there are two distinct regimes of coherent population transfer using transform-limited pulses: π pulse dynamics and adiabatic passage dynamics. Both methods can be used to create coherent superpositions starting from a particular pure state. Any method that prepares coherent superpositions must contain at least one stage sensitive to the parameters of excitation. In the π pulse regime, one creates coherent superpositions by controlling the pulse area. Complete population inversion can be achieved when the pulse area is an odd integer of π. In the adiabatic passage regime, one usually has complete population transfer to the target state as long as the adiabatic condition is satisfied. Partial population transfer can be achieved only for small values of the adiabatic parameter or, in some cases, by controlling the ratio between different Rabi frequencies [11–13].

In this Letter we propose a method to prepare entangled states by controlling the amplitude and relative phase in the fields used to excite the quantum system in the adiabatic regime. Essentially, we choose the relative phase as the control knob to create the desired target state. The suggested method opens a new avenue for quantum control of entanglement and facilitates experimental implementation of quantum logic gates.

Let us consider two interacting two-level quantum systems. As an example, such system can be two $1/2$ spin particles, two quantum dots, or two Rydberg atoms. Such quantum systems can be described as a four-level system in closed-loop configuration. We assume that each transition is driven separately by radio frequency pulses in the case of a spin system or by laser pulses for an atomic system.

The total wave function of the combined quantum system, $|\Psi(t)\rangle = a_1(t)|00\rangle + a_2(t)|11\rangle + b_1(t)|01\rangle + b_2(t)|10\rangle$, obeys the Schrödinger equation with the Hamiltonian in the rotating wave approximation (RWA) of the form

$$H = -\frac{1}{2}\begin{pmatrix} 0 & 0 & \Omega_{12}(t) & \Omega_{13}(t) \\ 0 & 0 & \Omega_{24}(t) & \Omega_{34}(t)e^{i\phi} \\ \Omega_{12}(t) & \Omega_{24}(t) & 2\Delta_1 & 0 \\ \Omega_{13}(t) & \Omega_{34}(t)e^{-i\phi} & 0 & 2\Delta \end{pmatrix}, \tag{1}$$

where $\Omega_{ij}(t)$ are the Rabi frequencies chosen to be real and $\Delta_{1,2}$ are one-photon detunings. Here we explicitly write the phase factor, $e^{-i\phi}$, where ϕ is the phase difference between Rabi frequencies. The origin of this phase can be traced to the dipole moments or to a single phase difference in the exciting fields [14]. We consider the case of two-photon resonance between states $|00\rangle$ and $|11\rangle$, and for simplicity we choose $\Omega_{12}(t) = \Omega_{13}(t) = \Omega_p(t)$ and $\Omega_{24}(t) = \Omega_{34}(t) = \Omega_s(t)$.

It is clear from the general structure of the Hamiltonian that there are two possible pathways from state $|00\rangle$ to state $|11\rangle$: $|00\rangle \xrightarrow{\Omega_{12}(t)} |01\rangle \xrightarrow{\Omega_{24}(t)} |11\rangle$ and $|00\rangle \xrightarrow{\Omega_{13}(t)} |10\rangle \xrightarrow{\Omega_{34}(t)} |11\rangle$. The dynamics and the role of the phase ϕ

on the final outcome of the population transfer in the closed-loop four-level system can ultimately be explained from interference between these two channels. In fact, the relative phase can be used as a control parameter in almost any general n-qubit system. We note some investigation of the phase-dependent dynamics in multilevel systems along these lines in [15–18]. We show below that by controlling the phase ϕ it is possible to create any entangled state of the form $|\Phi\rangle = (|00\rangle + e^{i\beta}|11\rangle)/\sqrt{2}$ and $|\Psi\rangle = (|01\rangle + e^{i\beta}|10\rangle)/\sqrt{2}$.

In this Letter we consider excitation by partially or half counterintuitive (HCI) pulse sequences when $\Omega_s(t)$ precedes $\Omega_p(t)$ and both of them turn off simultaneously. It is well known that such pulse sequences applied to the three-level λ system can be used to create coherent superpositions of ground and target states [11–13,19]. Here we demonstrate an implementation of this pulse sequence to create entangled states.

Figure 1 shows examples of population dynamics obtained by numerical solution of the Schrödinger equation with the Hamiltonian of Eq. (1) for the resonant excitation, $\Delta_{1,2} = 0$. We have chosen pulses with $\sin^4 t$ and $\cos^4 t$ shapes at the beginning and at the end, respectively. Using the HCI sequence, we create entangled states $|\Phi\rangle$, Fig. 1(a), or $|\Psi\rangle$, Fig. 1(b), depending only on the value of the relative phase ϕ. In addition, we present in Fig. 1(d) an example of state population at final time as a function of phase ϕ. The final populations imply oscillations between $|\Phi\rangle$ and $|\Psi\rangle$. Therefore, using a fixed pulse sequence we can prepare various entangled states just by tuning the relative phase between the pulses.

To understand the dynamics of population transfer and the role of the phase we consider the dressed states picture. In general, the eigenvalues of the Hamiltonian [Eq. (1)] have the form $\lambda_{1,2} = \mp\frac{1}{2}\lambda_-$, $\lambda_{3,4} = \mp\frac{1}{2}\lambda_+$, where $\lambda_\mp = \sqrt{\Omega_p^2(t) + \Omega_s^2(t) \mp \overline{\Omega}^2(t)}$, $\overline{\Omega}^2(t) = \sqrt{\Omega_p^4(t) + \Omega_s^4(t) + 2\cos\phi\,\Omega_p^2(t)\Omega_s^2(t)}$. The corresponding formulas for dressed states, $|c_i(t)\rangle$, are lengthy and will be presented elsewhere.

To determine which state is involved in the dynamics, we obtain expressions for the dressed states at particular

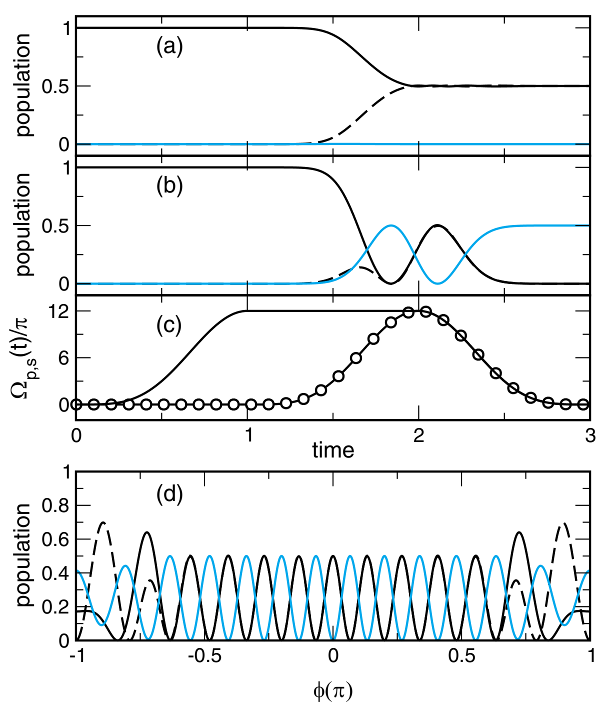

FIG. 1 (color online). Population dynamics at $\phi = 0$ (a) and $\phi = 0.2\pi$ (b); solid line, $|00\rangle$ state; dashed line, $|11\rangle$ state; light solid line, $|01\rangle$ and $|10\rangle$ states (coincide). (c) The HCI pulse sequence: $\Omega_s(t)$, solid line; $\Omega_p(t)$, solid line with circles. (d) Populations at final time vs phase with $S_p = \int \Omega_p(t)dt = 9\pi$ and $S_s = \int \Omega_s(t)dt = 21\pi$.

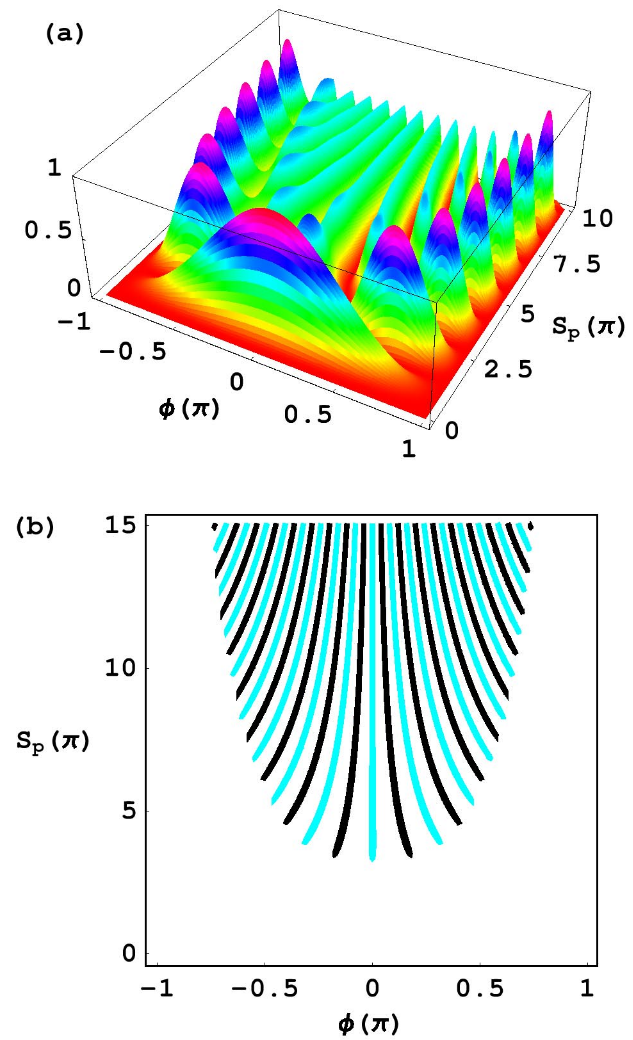

FIG. 2 (color online). (a) Final population of the $|11\rangle$ state as a function of the Rabi frequency and phase for the HCI pulse sequence, with relative pulse areas S_p: S_s = 3: 7. (b) Contour plot for the final population of the entangled states, $(|00\rangle - e^{i\phi/2}|11\rangle)/\sqrt{2}$ (dark solid line) and $(|01\rangle - e^{i\phi/2}|10\rangle)/\sqrt{2}$ (light solid line), as a function of the Rabi frequency and phase for the same HCI sequence.

limits. For the case of counterintuitive turn-on of the pulses, when $\Omega_s(t)$ precedes $\Omega_p(t)$, the initial state can be found by taking the limit $\Omega_p(t)/\Omega_s(t) \to 0$ in the general expressions of the dressed states. We obtain the following equations: $|c_1(0)\rangle \doteq (1/\sqrt{2}, 0, -ie^{i\phi/2}/2, ie^{-i\phi/2}/2)$, $|c_2(0)\rangle \doteq (1/\sqrt{2}, 0, ie^{i\phi/2}/2, -ie^{-i\phi/2}/2)$, $|c_3(0)\rangle \doteq (0, 1/\sqrt{2}, 1/2, e^{-i\phi}/2)$, $|c_4(0)\rangle \doteq (0, 1/\sqrt{2}, -1/2, -e^{-i\phi}/2)$. Therefore, initially the system is in the superposition of dressed states $[|c_1(0)\rangle + |c_2(0)\rangle]/\sqrt{2}$.

To find the final state of the system using the HCI pulse sequence we take the limit $\Omega_s(t)/\Omega_p(t) \to 1$. We obtain $|c_1(\infty)\rangle \doteq (e^{-i\phi/4}, -e^{i\phi/4}, -i, ie^{-i\phi/2})/2$, $|c_2(\infty)\rangle \doteq (e^{-i\phi/4}, -e^{i\phi/4}, i, -ie^{-i\phi/2})/2$, $|c_3(\infty)\rangle \doteq (e^{-i\phi/4}, e^{i\phi/4}, 1, e^{-i\phi/2})/2$, $|c_4(\infty)\rangle \doteq (e^{-i\phi/4}, e^{i\phi/4}, -1 - e^{-i\phi/2})/2$. From these expressions we conclude that in the adiabatic limit for the HCI sequence, we are able to prepare $|00\rangle - e^{i\phi/2}|11\rangle$ or $|01\rangle - e^{-i\phi/2}|10\rangle$, where the form and phase of the entangled states are controlled at will. Figure 2 shows the area of parameters where we can create these entangled states. The oscillation frequency between the entangled states [Fig. 1(d)] depends on the dynamical phase, $\int_0^\infty \lambda_-(t')dt'$, which is a function of the intensity and relative phase of the fields.

We see from Fig. 1(d) and Fig. 2 that for phases near $\phi \to \pm\pi$, all four states are populated and it is not possible to create an entangled state. The reason for this is the nonadiabatic couplings. Using the general expressions for the dressed states, it can be shown that the nonadiabatic couplings are zero except between the pairs of dressed states $|c_1(t)\rangle$, $|c_2(t)\rangle$ and $|c_3(t)\rangle$, $|c_4(t)\rangle$. These couplings become very important for phases near $\phi \to \pm\pi$. However, exactly at $\phi = \pm\pi$, it is again possible to create entangled states of the form $(|01\rangle + |10\rangle)/\sqrt{2}$ [the area of the parameters is not shown in Fig. 2(b)]. A detailed analysis of that particular case and nonadiabatic couplings will be presented elsewhere.

In the case of large detunings, $|\Delta_{1,2} \gg 0|$, we can adiabatically eliminate the probability amplitudes of the states $|01\rangle$ and $|10\rangle$ from the Schrödinger equation. We obtain the effective equation for the probability amplitudes $a_1(t)$ and $a_2(t)$ with Hamiltonian

$$H = \begin{pmatrix} \Omega_p^2(t) & \frac{\Omega_p(t)\Omega_s(t)}{4\Delta}(1 + e^{-i\phi}) \\ \frac{\Omega_p(t)\Omega_s(t)}{4\Delta}(1 + e^{i\phi}) & \frac{\Omega_s^2(t)}{2\Delta} \end{pmatrix}. \quad (2)$$

Here we use $\Delta_1 = \Delta_2 = \Delta$. It is interesting to see that the ac Stark shifts of the ground $|00\rangle$ and target $|11\rangle$ states are proportional to $\Omega_p^2(t)$ and $\Omega_s^2(t)$, respectively. The coupling between these states depends on phase. In particular, for $\phi = \pm\pi$, the coupling is exactly zero. Therefore, regardless of the field strength and two-photon resonance, the population remains in $|00\rangle$.

Using the eigenvalues of the Hamiltonian, $\lambda_{1,2} = [\Omega_p^2(t) + \Omega_s^2(t) \mp \overline{\Omega}^2(t)]/4\Delta$, we obtain the following dressed states

$$\begin{aligned} |c_1(t)\rangle &\doteq \left(\frac{\xi_+}{\overline{\Omega}(t)}, -\frac{e^{i\phi/2}\Omega_s(t)\Omega_p(t)\cos(\phi/2)}{\overline{\Omega}(t)\xi_+}\right), \\ |c_2(t)\rangle &\doteq \left(\frac{\xi_-}{\overline{\Omega}(t)}, \frac{e^{i\phi/2}\Omega_s(t)\Omega_p(t)\cos(\phi/2)}{\overline{\Omega}(t)\xi_-}\right), \end{aligned} \quad (3)$$

where $\xi_\pm = \sqrt{\overline{\Omega}^2(t) \pm (\Omega_s^2(t) - \Omega_p^2(t))}/\sqrt{2}$.

The limit $\Omega_p(t)/\Omega_s(t) \to 0$ gives $|c_1(0)\rangle \doteq (1, 0)$ and $|c_2(0)\rangle \doteq (0, e^{i\phi/2})$. Therefore, for counterintuitive turn on of the pulses, the system is initially in $|c_1(t)\rangle$. By taking the limit $\Omega_s(t)/\Omega_p(t) \to 0$, we obtain $|c_1(0)\rangle \doteq (0, -e^{i\phi/2})$, $|c_2(0)\rangle \doteq (1, 0)$, so that the system is initially in $|c_2(0)\rangle$ for intuitive turn-on.

The general expression for nonadiabatic couplings in this case is very simple: $U_N(t) = \sigma_y \dot{\Omega}(t)(\Omega_p^2(t) + \Omega_s^2(t))\cos(\phi/2)/\overline{\Omega}^4(t)$, where σ_y is the Pauli matrix and $\dot{\Omega}(t) = \dot{\Omega}_p(t)\Omega_s(t) - \Omega_p(t)\dot{\Omega}_s(t)$. Our analysis shows that the condition of adiabaticity is very similar to the Stimulated Raman Adiabatic Passage (STIRAP) case [20], so that we can neglect nonadiabatic coupling for strong laser pulses. Assuming the adiabatic approximation, we can predict the state of the system at a final time. The limit $\Omega_s(t)/\Omega_p(t) \to 1$ gives $|c_1(\infty)\rangle \doteq (1, -e^{i\phi/2})/\sqrt{2}$ and $|c_2(\infty)\rangle \doteq (1, e^{i\phi/2})/\sqrt{2}$. This implies that we can prepare the entangled state $(|00\rangle - e^{i\phi/2}|11\rangle)/\sqrt{2}$ using the HCI pulse sequence and the entangled state $(|00\rangle + e^{i\phi/2}|11\rangle)/\sqrt{2}$ using a partially or half intuitive (HI) pulse sequence.

Numerical simulations confirm our findings from the dressed state picture. Figure 3 shows the final population of the $|11\rangle$ state as a function of phase and the area of the $\Omega_p(t)$ pulse. The results have been obtained for the HCI sequence. There is a large area in the parameter space where 50% of population is transferred to $|11\rangle$. This corresponds to preparing the entangled state

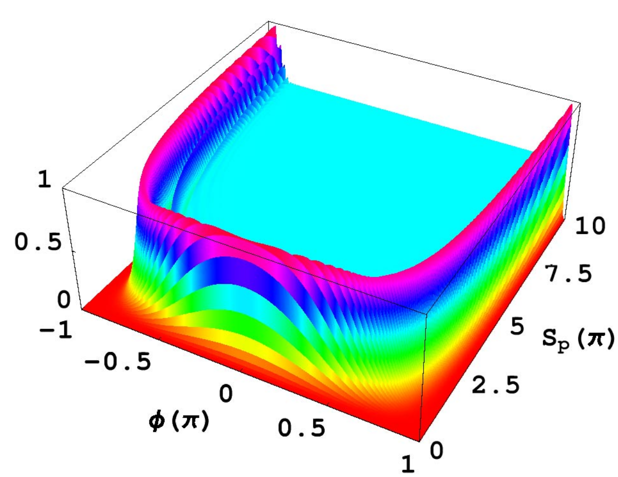

FIG. 3 (color online). Final population of the $|11\rangle$ state as a function of the Rabi frequency and phase for the off-resonant excitation by the HCI pulse sequence.

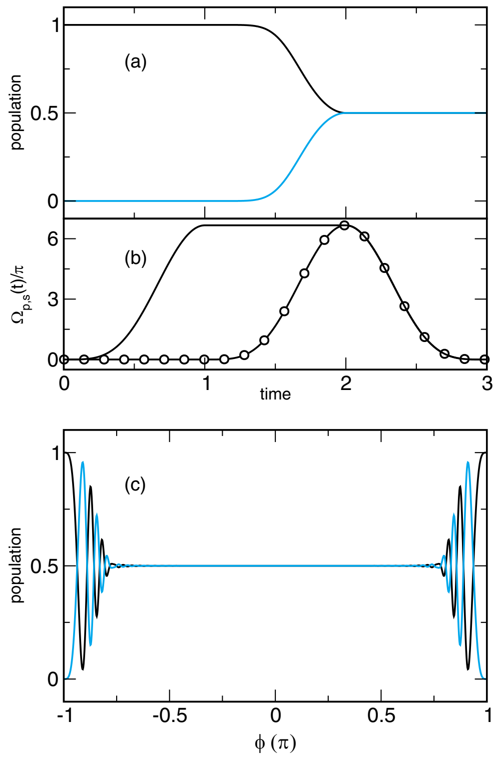

FIG. 4 (color online). (a) Population dynamics of the state $|00\rangle$ (dark line) and $|11\rangle$ (light line) at $S_p = 5\pi$. (b) The HCI pulse sequence: $\Omega_s(t)$, solid line; and $\Omega_p(t)$, solid line with circles. (c) Populations at final time vs phase with $S_p = 5\pi$; S_p: $S_s = 3$: 7.

$(|00\rangle - e^{i\phi/2}|11\rangle)/\sqrt{2}$. At $\phi = \pm\pi$, the population is locked in the ground state since the effective Rabi frequency, $\Omega_p(t)\Omega_s(t)(1 + e^{-i\phi})/(2\Delta)$, is equal to zero. Population dynamics for the off-resonant excitation are shown in Fig. 4. Since the effective Rabi frequency becomes smaller as $\phi \rightarrow \pm\pi$, the nonadiabatic couplings become essential for larger values of phase, where it is more difficult to prepare entangled states, as shown in Fig. 4(c).

In summary, we have demonstrated the possibility of creating entangled states by controlling the relative phase of the external fields in a composite system of two interacting two-level quantum systems. We have indicated the direct relationship between the controlled relative phase and the prepared phase of the entanglement. In the resonant scheme, the results show an oscillatory behavior of the populations of the entangled states as a function of the relative phase at fixed envelopes of the Rabi frequencies, while they are insensitive either to the phase and to the Rabi frequencies in the off-resonant case.

Experimentally, the most challenging part of both schemes is the turn-off stage of the pulses, when the ratio between four Rabi frequencies should be fixed. In fact, in the off-resonant scheme, the requirements are also simpler to meet, since it is only necessary to make $\Omega_{12}(t)\Omega_{13}(t) = \Omega_{24}(t)\Omega_{34}(t)$ at final time. For some two interacting two-level systems there are clear correlations between the pair of transitions, which simplifies the operational setup and provides robustness of the scheme to pulse fluctuations. With recent progress in pulse shaping techniques [21], several alternatives make feasible the experimental implementation of the schemes.

The scalability of the proposed method is an open question that will be addressed elsewhere. We believe that if the specific couplings of N interacting quantum systems allows to find any closed-loop structure in the corresponding 2^N-level system, then control of the population (read entanglement) can be achieved by controlling a relative phase of the external fields. The simplicity of the phase adjustment and the relative robustness of the schemes against moderate changes of the pulse parameters is an advantage of the proposed methods.

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