Efficient Use of Quantum Resources for the Transmission of a Reference Frame

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We propose a covariant protocol for transmitting reference frames encoded on N spins, achieving sensitivity N^{-2} without the need of a preestablished reference frame and without using entanglement between sender and receiver. The protocol exploits the use of equivalent representations that were overlooked in the previous literature.

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In the ideal world of classical physics, spatial directions and reference frames can be communicated with arbitrary accuracy using classical communication and a preestablished common frame, or by just sending physical objects, such as gyroscopes. In the second case, if Alice wants to send a frame to Bob, she needs only to align the rotation axes of her gyroscopes with the directions she wants to communicate before sending them to Bob. Once Bob has measured the direction of the gyroscopes, a common reference frame has been established. Clearly, in the real world, arbitrary accuracy is limited by quantum fluctuations. However, similarly to the case of phase estimation [1,2], we can learn how to harness the quantum laws in order to achieve the ultimate precision limits of the communication protocol.

The primitive systems that one can use for communication of reference frames are quantum spins, since they can be considered as elementary quantum gyroscopes. In this scenario, Alice transmits a Cartesian reference frame by preparing N spins in a quantum state $|A\rangle$ which is related to her set of Cartesian axes $\mathbf{n}^{(A)} \doteq \{n_x^A, n_y^A, n_z^A\}$ and by sending them to Bob. With respect to Bob's axes $\mathbf{n}^{(B)} \doteq \{n_x^B, n_y^B, n_z^B\}$, such a state corresponds to $|A_g\rangle \doteq U_g^{\otimes N}|A\rangle$, where the unitary matrix U_g represents the rotation g connecting Bob's frame to Alice's one, namely $\mathbf{n}^{(A)} = g\mathbf{n}^{(B)}$. Now, Bob's task is to estimate the rotation g of the state $|A_g\rangle$, and then to align his axes with Alice's frame. It is worth noting that such a scheme works without the need of any preestablished reference frame. Notice also that the problem of aligning reference frames using quantum spins is formally equivalent to the problem of estimating unknown $\mathbb{SU}(2)$ rotations (which is the same problem of estimating the dynamics of an unknown qubit gate [3–5]).

For the estimation of rotations with a finite number N of spins there is a nonzero probability of error which vanishes in the limit of infinite N. Now the issue is to optimize the accuracy of the estimation for a given N by properly choosing Bob's measurement and Alice's input state $|A\rangle$. In the recent literature [6–9], much progress has been made in this direction, and a number of strategies have been proposed in specific cases. Nevertheless, in some of these works [7,8], it was argued that equivalent representations of $\mathbb{SU}(2)$ are redundant for encoding rotations, and this oversight led to false claims of optimality in Ref. [7], where an asymptotic average error $1/N$ was found. In this Letter, we show that, on the contrary, equivalent representations play a crucial role in enhancing the sensitivity of the estimation, since the inclusion of multiple equivalent representations increases the dimension of the Hilbert space available to storing information. Moreover, we resolve a long-standing controversy over whether the optimal strategy is covariant or not. In Ref. [8], a noncovariant strategy is shown to do better (with an error scaling as $1/N^2$) than the covariant strategy in Ref. [7]. While the latter strategy was mistakenly thought to be best, it appeared that the best covariant strategy was not optimal. The present Letter resolves the puzzle by showing that the optimal covariant strategy does just as well as those presented in Ref. [8] with an asymptotic error $1/N^2$.

Finally, as we will show, there is a relation between the present scheme and the entangled protocol of Ref. [5], with the role of entanglement here played by equivalent representations.

Let us now summarize the main points in the problem of estimating $\mathbb{SU}(2)$ rotations. The most general estimation strategy that Bob can perform—including both measurements and data analysis—is described by a positive operator-valued measure (shortly POVM), namely, by a set of positive operators $\{M(g)\}$ in the Hilbert space of N spins such that $\int dg M(g) = \mathbb{1}$, with the integral extended to the whole $\mathbb{SU}(2)$ group, and dg denoting the invariant Haar measure on $\mathbb{SU}(2)$, normalized such that $\int dg = 1$. The probability density of estimating g when the true rotation is g_* is given by the Born rule: $p(g|g_*) \doteq \mathrm{Tr}[M(g)|A_{g_*}\rangle\langle A_{g_*}|]$. Finally, the efficiency of a strategy is defined in terms of the transmission error

$$e(g, g_*) \doteq \sum_{\alpha=x,y,z} |g n_\alpha^B - g_* n_\alpha^B|^2, \qquad (1)$$

which quantifies the deviation between the estimated axes

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and the true ones. The maximization of the efficiency then corresponds to the minimization of the average error

$$\langle e\rangle = \int dg_* \int dg p(g|g_*)e(g, g_*). \tag{2}$$

Notice that we have assumed a uniform *a priori* distribution dg_* for the true rotations, according to the fact that g_* is completely unknown. Since the function $e(g, g_*)$ enjoys the invariance property $e(g, g_*) = e(hg, hg_*)$ for any $h \in \mathbb{SU}(2)$, as proved by Holevo [1], there is no loss of generality in assuming that Bob's strategy is described by a *covariant* POVM, namely

$$M(g) \doteq U_g^{\otimes N} \Xi U_g^{\dagger\otimes N}, \tag{3}$$

with Ξ a positive operator. This fact relies on the covariance of the set of input states. Indeed, for an arbitrary POVM $N(g)$ one can always construct a covariant one with the same average error, corresponding to $\Xi \doteq \int dg U_g^{\dagger\otimes N} N(g) U_g^{\otimes N}$.

Let us now enter the core of our method. In what follows, our aim will be to use equivalent representations for constructing a highly efficient reference state $|A\rangle$ in the space $\mathsf{H}^{\otimes N}$ of N spins. For this purpose, $\mathsf{H}^{\otimes N}$ can be conveniently decomposed in terms of the Clebsch-Gordan series, i.e., as direct sum of orthogonal subspaces which are irreducible under the action of $\mathbb{SU}(2)$ rotations, namely

$$\mathsf{H}^{\otimes N} = \bigoplus_{j=0(\frac{1}{2})}^{J} \bigoplus_{\alpha=1}^{n_j} \mathsf{H}_{j\alpha}. \tag{4}$$

Here, j represents, as usual, the quantum number of the total angular momentum: it runs from 0 ($\frac{1}{2}$) to $J = \frac{N}{2}$ for N even (odd), and labels the equivalence class of each irreducible representation. On the other hand, α is a degeneracy index labeling different equivalent representations in the same class j. For example, with three spins, one has $\mathbf{\frac{1}{2}^{\otimes 3}} = \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}$, so that for the class $j = \frac{1}{2}$, there are two equivalent irreducible representations corresponding to two orthogonal subspaces. The number n_j of equivalent representations in the class j is given by [10]

$$n_j = \frac{2j+1}{J+j+1}\binom{2J}{J+j}. \tag{5}$$

In each invariant subspace $\mathsf{H}_{j\alpha}$, we can introduce the basis $\{|j\alpha, m\rangle; m = -j, \ldots, j\}$ made of eigenvectors of the z component of the total angular momentum. With respect to these bases, $\mathbb{SU}(2)$ rotations are represented by the ordinary Wigner matrices $U_{nm}^{(j)}(g)$, namely

$$U_g^{\otimes N}|j\alpha, m\rangle = \sum_{n=-j}^{j} U_{nm}^{(j)}(g)|j\alpha, n\rangle. \tag{6}$$

Notice that two vectors $|j\alpha, m\rangle$ and $|j\beta, m\rangle$ belonging to different orthogonal subspaces $\mathsf{H}_{j\alpha}$ and $\mathsf{H}_{j\beta}$ transform in the same way under $\mathbb{SU}(2)$ rotations. Let us define, then, the operator

$$T_{\alpha\beta}^{(j)} \doteq \sum_{m=-j}^{j} |j\alpha, m\rangle\langle j\beta, m| \tag{7}$$

that takes a vector in the space $\mathsf{H}_{j\beta}$ to the corresponding one in $\mathsf{H}_{j\alpha}$. Using this operator, we will compare vectors in different equivalent subspaces, and we will say that two vectors $|\psi_{j\alpha}\rangle \in \mathsf{H}_{j\alpha}$ and $|\varphi_{j\beta}\rangle \in \mathsf{H}_{j\beta}$ are *iso-orthogonal* if $\langle\psi_{j\alpha}|T_{\alpha\beta}^{(j)}|\varphi_{j\beta}\rangle = 0$.

As opposite to the approach used in the previous works, here the state $|A\rangle$ will be chosen in order to use as many equivalent representations as possible. For this purpose, the crucial point is that the maximum number of representations one can exploit in the class j is not n_j, but $k_j \doteq \min\{n_j, 2j+1\}$, corresponding to the fact that equivalent representations are useful only when one takes iso-orthogonal vectors in different representations. The proof of this statement has been derived in [11] and relies on the fact that for any given vector $|A\rangle$, there is always a rearrangement of the decomposition (4) such that $|A\rangle$ has components on at most k_j representations from the class j, and these components are all iso-orthogonal to each other. Using (5), it is easy to see that $k_j = 2j+1$ for $j < J$ and $k_J = n_J = 1$. Keeping this in mind, we make the following choice for Alice's reference vector

$$|A\rangle = A_J|J, J\rangle + \sum_{j=0(\frac{1}{2})}^{J-1} \sum_{\alpha=1}^{2j+1} \frac{A_j}{\sqrt{2j+1}}|j\alpha, m(\alpha)\rangle, \tag{8}$$

where without loss of generality, $A_j \geq 0$, and $m(\alpha)$ is an injective function, namely $m(\alpha) \neq m(\alpha')$ if $\alpha \neq \alpha'$, according to the idea of taking an iso-orthogonal vector for each equivalent representation. Notice that the term for $j = J$, which has multiplicity $n_J = 1$, has been chosen arbitrarily with $m = J$. However, as we will see in the following, its contribution is negligible in the asymptotic limit of large N.

Now we need to specify which covariant POVM Bob must use to extract the rotation g from the state $|A_g\rangle$; namely, we must provide the operator Ξ in Eq. (3). First, we observe that, since the vector $|A\rangle$ lies in the invariant subspace of $\mathsf{H}^{\otimes N}$

$$\mathsf{K} = \mathsf{H}_J \oplus \bigoplus_{j=0(\frac{1}{2})}^{J-1} \bigoplus_{\alpha=1}^{2j+1} \mathsf{H}_{j\alpha}, \tag{9}$$

the probability distribution

$$p(g|g_*) = \langle A_{g_*}|U_g^{\otimes N} \Xi U_g^{\dagger\otimes N}|A_{g_*}\rangle \tag{10}$$

depends only on the restriction $\xi \doteq P\Xi P$, where P is the projection on K. Second, instead of optimizing Bob's POVM in order to minimize the transmission error (2), here we will take the *maximum likelihood POVM* [11], namely, the POVM which maximizes the peak $p(g_*|g_*)$ in the probability distribution $p(g|g_*)$. For this POVM, one simply has $\xi = |B\rangle\langle B|$, where

$$|B\rangle = \sqrt{2J+1}|JJ\rangle + \sum_{j=0(\frac{1}{2})}^{J-1}\sum_{\alpha=1}^{2j+1}\sqrt{2j+1}|j\alpha, m(\alpha)\rangle. \quad (11)$$

We stress that in the eigenstates of Eq. (11), the z component of the total angular momentum is referred to Bob's axes, hence the transmission protocol does not require a common reference frame (we remind that Alice's state $|A\rangle$ is seen as $|A_g\rangle = U_g^{\otimes N}|A\rangle$ in Bob's reference frame).

With the previous settings, the problem of optimizing the coefficients $\{A_j\}$ in the state $|A\rangle$ in order to minimize the transmission error becomes straightforward. First, one can note [7] that $e(g|g_*) = 6 - 2\chi(gg^{*-1})$, where $\chi(g) \doteq \sum_{m=-1}^{1} U_{mm}^{(1)}(g)$ is the character of the Wigner matrices for $j = 1$. Then, minimizing the average error $\langle e\rangle$ is equivalent to maximizing the average character

$$\langle\chi\rangle \doteq \int dg\chi(g)p(g|e); \quad (12)$$

e denoting the identical rotation. Notice that the integral over g_* in (2) has been performed by exploiting the invariance property of covariant POVM's, i.e., $p(g|g_*) = p(hg|hg_*)$, $\forall h \in \mathbb{SU}(2)$. Using the identity

$$\int dg U_{mm}^{(1)}(g)U_{rs}^{(j)}(g)U_{ik}^{(l)*}(g) = \frac{1}{2l+1}\langle 1mjr|li\rangle\langle lk|1mjs\rangle, \quad (13)$$

where $\langle 1mjr|li\rangle$ denote the Clebsch-Gordan coefficients, and performing the sums over equivalent representations, we obtain

$$\langle\chi\rangle = \sum_{j,l=0(\frac{1}{2})}^{J} A_j M_{jl} A_l \equiv A^T \mathsf{M} A, \quad (14)$$

where A denotes the column vector $(A_J, A_{J-1}, \ldots, A_{0(1/2)})$ and M is the tridiagonal matrix

$$\mathsf{M} \doteq \begin{pmatrix} \frac{J}{J+1} & \frac{1}{\sqrt{2J+1}} & & & & & \\ \frac{1}{\sqrt{2J+1}} & 1 & 1 & & & 0 & \\ & 1 & 1 & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & & & \\ & 0 & & & 1 & 1 & 1 \\ & & & & & 1 & \zeta \end{pmatrix}. \quad (15)$$

Here, $\zeta = 0(1)$ for even (odd) values of N. Since the normalization of Alice's vector implies $A^T A = 1$, maximizing $\langle\chi\rangle$ simply consists in finding the greatest eigenvalue λ for the matrix M: λ is actually the maximum $\langle\chi\rangle$ for our strategy and the optimal coefficients $\{A_j\}$ are the components of the corresponding normalized eigenvector.

For small N, one can easily perform numerical diagonalization:, for example, with $N = 3$, 5, and 9, one finds $\lambda = 1.3886$, 2.0864, and 2.6294, respectively. These values can be compared with those obtained in Ref. [7] without the use of equivalent representations: even for $N = 3$, one can see a 17% improvement of $\langle\chi\rangle$. On the other hand, in the asymptotic limit of large N, an analytical treatment is possible, which is essentially based on the fact that the contribution of the J representation becomes negligible. Let us denote the dependence on N by writing $\mathsf{M}^{(N)}$ and $\lambda^{(N)}$. If we introduce the matrix $\mathsf{T}^{(N)}$ obtained from $\mathsf{M}^{(N)}$ by canceling the first row and the first column (corresponding to ignore the J representation) and call $\sigma^{(N)}$ its greatest eigenvalue, then we have $\lambda^{(N)} \geq \sigma^{(N)}$. Nevertheless, it is also easy to see that $\sigma^{(N+2)} \geq \lambda^{(N)}$, due to the fact that $0 \leq \mathsf{M}_{ij}^{(N)} \leq \mathsf{T}_{ij}^{(N+2)}$ for any i, j [12]. Hence, the asymptotic behavior of $\lambda^{(N)}$ is bounded by $\sigma^{(N)} \leq \lambda^{(N)} \leq \sigma^{(N+2)}$. The matrix $T^{(N)}$ can be analytically diagonalized in terms of Chebyshev polynomials, and its greatest eigenvalue is $\sigma^{(N)} = 1 + 2\cos(\frac{2\pi}{N+1})$. This implies the asymptotic behavior $\langle\chi\rangle \sim 3 - \frac{4\pi^2}{N^2}$, corresponding to the following power law for the transmission error

$$\langle e\rangle \sim \frac{8\pi^2}{N^2}. \quad (16)$$

Comparing this result with the behavior $\langle e\rangle \sim \frac{8}{N}$ of [7], one can observe a quadratic improvement due to the use of equivalent representations.

Notice that $\langle e\rangle \sim \frac{8\pi^2}{N^2}$ is also the same efficiency of the protocol in [9], where, by adopting the idea introduced in Ref. [5], entanglement between sender and receiver is exploited, and a collective measurement on two sets of N spins is performed. With respect to such protocol, the present scheme provides a saving of resources (i.e., half number of spins and no need of entanglement between Alice and Bob), and, more importantly, does not require a preestablished reference frame [13].

There exists a connection between the present protocol and the entanglement-assisted one. In fact, let us introduce for any class the *representation space* H_j of dimension $2j + 1$ and the *multiplicity space* M_j of dimension n_j, and write $|j\alpha, m\rangle$ as $|jm\rangle \otimes |\alpha\rangle \in \mathsf{H}_j \otimes \mathsf{M}_j$. Choosing $\{|\alpha\rangle; \alpha = 1, \ldots, n_j\}$ as an orthonormal basis for M_j, one has

$$\bigoplus_{\alpha=1}^{n_j} \mathsf{H}_{j\alpha} \equiv \mathsf{H}_j \otimes \mathsf{M}_j. \quad (17)$$

By means of such isomorphism, we can rewrite our choice of Alice's state as

$$|A\rangle \equiv A_J|JJ\rangle + \sum_{j=0(\frac{1}{2})}^{J-1} A_j|E_j\rangle, \qquad (18)$$

where

$$|E_j\rangle \doteq \frac{1}{\sqrt{2j+1}} \sum_{\alpha=1}^{2j+1} |jm(\alpha)\rangle \otimes |\alpha\rangle \qquad (19)$$

is a maximally entangled state between the representation space H_j and the multiplicity space M_j [14]. If we neglect the J term in $|A\rangle$, then we get a vector which is formally the same as in [9]. This means that the protocol exploiting entanglement and $2N$ spins is reproduced using N spins and without entanglement between sender and receiver. We stress that here the entanglement is between the representation and the multiplicity space (which is not necessarily related to entanglement between the N physical spins).

In conclusion, in this Letter, we have shown how to exploit equivalent representations of the rotation group for saving quantum resources in transmitting a reference frame. A quadratic improvement of the transmission efficiency has been achieved with respect to the protocol of Ref. [7] which mistakenly neglects equivalent representations. This is due to the fact that the use of such representations provides more room for storing information. An intuitive justification of this fact is provided by the maximum likelihood strategy [11]: in fact, the maximum likelihood for a pure state is exactly proportional to the dimension of its orbit under the action of the group, and for N spins, this is at most $d_{\max} = (2J+1) + \sum_{i=0(1/2)}^{J-1}(2j+1)^2 \sim N^3$. In our protocol, this dimension is fully exploited by entangling the representation space with the multiplicity space, whereas without such entanglement one would obtain a dimension $d = (2J+1) + \sum_{j=0(1/2)}^{J-1}(2j+1) \sim N^2$. Notice that the use of multiplicity spaces has been found to be necessary also in optimal schemes for the transmission of elements of the permutation group [15], and in achieving the optimal capacity for private classical communication using a private shared reference frame [16].

Our results finally settle the controversy about covariance of the optimal protocol, which was raised in Ref. [8], by providing a covariant scheme with the same performance $1/N^2$.

We also proved how the presence of equivalent representations provides the remarkable possibility of reproducing the same efficiency of covariant entangled protocols without the need of a preestablished reference frame and without using entanglement between sender and receiver. The present use of equivalent representations is a general method which is not restricted to the transmission of reference frames, and is expected to provide useful improvements also in other estimation problems.

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[12] We compare N with $N+2$ terms, such that the considered matrices have the same value of the border term ζ.

[13] More precisely, the protocol in Ref. [9] requires that the sender and receiver have a finite resource of entanglement wherein the eigenvectors of their reduced density operators encode the orientations of their respective Cartesian frames (this can be established, for instance, if the sender and receiver set up the entanglement when their Cartesian frames are still aligned and lock the orientation of their particles to that of their lab as the relative orientation of their labs becomes unknown). Such a resource can only act as an *imperfect* shared reference frame. It does not provide an isomorphism between the Hilbert spaces of the sender and receiver, and does not allow any single-shot measurement of spin to be perfectly predictable.

[14] Notice that the property of iso-orthogonality becomes in the decomposition (17) the customary orthogonality in the representation space $\mathsf{H}j$.

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