Hamiltonian Description of Low-Temperature Relativistic Plasmas

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We develop a low-temperature fluidlike plasma model without recourse to a collisional closure. The equations are closed by treating the momentum spread asymptotically. This model inherits the Hamiltonian structure, including Casimir invariants of the Vlasov–Maxwell theory. We study temperature evolution in the wake of an intense laser pulse propagating in a plasma. We show that the momentum spread is intrinsically anisotropic and that, for conditions corresponding to recent experiments, modest heating occurs.

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The transition from a kinetic (phase-space) description to a moment based description is a well-known approach, with an equally well-known drawback of the closure problem. In ordinary gases, in which collective effects are the product of two- and three-body collisions, the Chapman–Enskog procedure provides the transport equations that close the moment equations [1]. In a collisional plasma, similar reasoning leads to fluidlike models [2]. In the limit of small collisionality, it is often assumed that meaningful "fluid" models can not be constructed due to the lack of a collisional closure. A simple counterexample to this view is the common cold-fluid model, where the distribution function $f(\mathbf{r}, \mathbf{p}, t) = n(\mathbf{r}, t)\delta(\mathbf{p} - \mathbf{P}(\mathbf{r}, t))$ with density n and hydrodynamic momentum $\mathbf{P}$. The resulting equations for n and $\mathbf{P}$ are naturally closed and represent an exact solution to the underlying kinetic equations.

Here we present a warm fluid model of a collisionless, relativistic plasma valid in the regime of small momentum spread. We perform an asymptotic expansion in the momentum-space width of the distribution function, keeping lowest order terms. In the absence of collisions, the pressure is not forced to be isotropic and new phenomena result. We apply this model to intense, short-pulse laser-plasma interactions and show that the momentum spread is highly anisotropic and that modest heating occurs. Proper characterization of the electron phase space is critical for applications such as high-gradient laser-plasma accelerators in which plasma heating strongly affects particle trapping and wave breaking.

The current generation of experiments (c.f. Refs. [3,4]) on the interaction of intense, short laser pulses with under-dense plasmas accesses a novel physical regime wherein the plasma electrons experience relativistic motion while the "temperature" (more properly, the momentum spread) is quite small, and collisions are practically nonexistent. This regime, which has been largely unexplored, stands in stark distinction to the usual setting for relativistic kinetic theories and other relativistic fluid models [5–9], where the plasma is assumed to have a relativistic *thermal velocity* and to be collisionally dominated. Such models are inappropriate for the short-pulse case since collisions rates are orders of magnitude too small to guarantee local thermodynamic equilibrium.

The relativistic Vlasov distribution function, f, is a Lorentz scalar [8]; thus we can write $f(x^\mu, p^\mu) = F(p^\mu \mathcal{A}'_\mu, p^\mu p^\nu \mathcal{B}'_{\mu\nu}, \ldots)$ where $\mathcal{A}'$, $\mathcal{B}'$, etc., can depend on $x^\mu = (ct, \mathbf{r})$ and $p^\mu = (\gamma mc, \mathbf{p})$ is the four momentum. In the general case this form is not particularly useful as one is forced to keep all terms. We can make significant progress under the assumption that the momentum spread is small. By small momentum spread we mean that f is only significant for $|(p_\mu - P_\mu)(p^\mu - P^\mu)| \ll m^2c^2$ where $P^\mu = (\gamma mc, \mathbf{P})$ is some characteristic momentum which we will define more concretely below. Converting to a $3 + 1$ formulation, this assumption on the momentum spread results in the distribution function taking the form $f(\mathbf{r}, \mathbf{p}, t) = F(\delta p_i \mathcal{A}_i, \delta p_i \delta p_j \mathcal{B}_{ij}, \ldots)$ where, $\mathcal{A}_i$, $\mathcal{B}_{ij}$, ...depend on space and time and $\delta p_i = p_i - P_i$. In more than one dimension, surfaces of constant $\delta p_i \mathcal{A}_i$ are not closed and so the distribution can not be a function of $\delta p_i \mathcal{A}_i$ alone. Thus the first viable truncation is at the quadratic term. In the following we will assume that the momentum dependence of f is through $Q = \delta p_i \delta p_j \mathcal{B}_{ij}$, where $\det\mathcal{B} \neq 0$. Under this assumption, the distribution function takes on the general form

$$f(\mathbf{r}, \mathbf{p}, t) = n(\mathbf{r}, t)\sqrt{\det\mathcal{B}}\, g(Q), \tag{1}$$

where n is the spatial density and g is some positive semidefinite integrable function normalized such that $4\pi \int ds s^2 g(s^2) = 1$. The dynamics of f in phase space is entirely determined by the dynamics of n, $\mathbf{P}$ and $\mathcal{B}$ in configuration space. The assumption of small momentum spread has allowed us to parameterize the distribution function and reduce the dynamics from six-dimensional phase-space to three-dimensional configuration space in much the same way that the cold-fluid reduction is achieved.

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We now turn to the problem of determining the dynamical equations for n, $\mathbf{P}$ and $\mathcal{B}$. The most expedient route to this end is to appeal to the noncanonical Hamiltonian structure of the Maxwell–Vlasov system [10]. This approach has the added benefit of guaranteeing the reduced model will possess important conservation laws such as energy and entropy. We begin by examining the bracket and then turn our attention to the Hamiltonian.

We can view our reduction (1) as a coordinate transformation where the inverse can be expressed in terms of moments of the distribution. One easily sees that

$$n(\mathbf{r}, t) = \int d^3\mathbf{p} f(\mathbf{r}, \mathbf{p}, t), \tag{2a}$$

$$\mathbf{P}(\mathbf{r}, t) = \frac{1}{n}\int d^3\mathbf{p}\mathbf{p} f(\mathbf{r}, \mathbf{p}, t), \tag{2b}$$

(from which we immediately see $\mathbf{P}$ is the average momentum). It will prove more convenient to introduce

$$\Pi_{ij}(\mathbf{r}, t) = \frac{1}{n}\int d^3\mathbf{p}\,\delta p_i \delta p_j f(\mathbf{r}, \mathbf{p}, t) \tag{2c}$$

in favor of $\mathcal{B}$. Direct calculation gives $\Pi_{ij} = \mathcal{B}^{-1}_{ij}\Delta_2$, where $\Delta_2 = (4\pi/3)\int ds s^4 g(s^2)$.

The moments (2) and the function g provide a complete characterization of the distribution function. Given g, for any functional $\tilde{F}[f]$, there exists a corresponding functional $F[n, \mathbf{P}, \Pi]$ such that $\tilde{F}[f] = F[n, \mathbf{P}, \Pi]$. Thus, consideration of functionals of n, $\mathbf{P}$ and Π formally includes all functionals of f. The chain rule gives

$$\frac{\delta \tilde{F}}{\delta f} = \frac{\delta F}{\delta n} + \frac{\delta F}{\delta P_i}\frac{\delta p_i}{n} + \frac{\delta F}{\delta \Pi_{(ij)}}\frac{\delta p_i \delta p_j - \Pi_{ij}}{n}, \tag{3}$$

where the parentheses on the subscripts indicates symmetrization of the functional derivative in the usual way: $\delta F/\delta\Pi_{(ij)} = (1/2)(\delta F/\delta\Pi_{ij} + \delta F/\delta\Pi_{ji})$. The Vlasov–Maxwell Poisson bracket is given by [10]

$$\{\tilde{F}, \tilde{G}\}_{\mathrm{V}} = \int d^3\mathbf{r} d^3\mathbf{p} f\left[\frac{\delta \tilde{F}}{\delta f}, \frac{\delta \tilde{G}}{\delta f}\right] + 4\pi q\int d^3\mathbf{r} d^3\mathbf{p}\nabla_p f\cdot\left[\frac{\delta \tilde{F}}{\delta \mathbf{E}}\frac{\delta \tilde{G}}{\delta f} - \frac{\delta \tilde{G}}{\delta \mathbf{E}}\frac{\delta \tilde{F}}{\delta f}\right] + 4\pi c\int d^3\mathbf{r}\left(\frac{\delta \tilde{F}}{\delta \mathbf{E}}\cdot\nabla\times\frac{\delta \tilde{G}}{\delta \mathbf{B}} - \frac{\delta \tilde{G}}{\delta \mathbf{E}}\cdot\nabla\times\frac{\delta \tilde{F}}{\delta \mathbf{B}}\right) + \frac{q}{c}\int d^3\mathbf{r} d^3\mathbf{p} f\mathbf{B}\cdot\nabla_p\frac{\delta \tilde{F}}{\delta f}\times\nabla_p\frac{\delta \tilde{G}}{\delta f}. \tag{4}$$

For our chosen form for f, the third-order moments vanish identically and we can obtain an exact expression for the action of the Poisson bracket on functionals of n, $\mathbf{P}$ and Π without specifying further details about g. Inserting (3) into (4) yields

$$\{F, G\}_M = \int d^3\mathbf{r}\bigg[\frac{\delta G}{\delta P_k}\partial_k\frac{\delta F}{\delta n} - \frac{\delta F}{\delta P_k}\partial_k\frac{\delta G}{\delta n} + \frac{1}{n}\left(\frac{\delta F}{\delta P_k}\frac{\delta G}{\delta \Pi_{(rs)}} - \frac{\delta G}{\delta P_k}\frac{\delta F}{\delta \Pi_{(rs)}}\right)\partial_k\Pi_{rs} + \frac{1}{n}\left(\frac{\delta F}{\delta P_k}\frac{\delta G}{\delta P_l} + 4\Pi_{rs}\frac{\delta F}{\delta \Pi_{(kr)}}\frac{\delta G}{\Pi_{(ls)}}\right) \times(\partial_k \mathfrak{p}_l - \partial_l \mathfrak{p}_k) + 2\Pi_{rs}\left(\frac{\delta G}{\Pi_{(rk)}}\partial_k\frac{1}{n}\frac{\delta F}{\delta P_s} - \frac{\delta F}{\delta \Pi_{(rk)}}\partial_k\frac{1}{n}\frac{\delta G}{\delta P_s}\right)\bigg] + 4\pi q\int d^3\mathbf{r}\left(\frac{\delta F}{\delta \mathbf{P}}\cdot\frac{\delta G}{\delta \mathbf{E}} - \frac{\delta G}{\delta \mathbf{P}}\cdot\frac{\delta F}{\delta \mathbf{E}}\right) + 4\pi c\int d^3\mathbf{r}\left(\frac{\delta F}{\delta \mathbf{E}}\cdot\nabla\times\frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{E}}\cdot\nabla\times\frac{\delta F}{\delta \mathbf{B}}\right), \tag{5}$$

where $\mathfrak{p}_i = P_i + qA_i/c$ is the canonical momentum and $\mathbf{A}$ is the vector potential. This result is *exact*: for any functionals $F[n, \mathbf{P}, \Pi]$ and $G[n, \mathbf{P}, \Pi]$ we have $\{F, G\}_M = \{F, G\}_V$. As a consequence we see that the moment bracket inherits the Jacobi identity (as well as all other properties) from the full bracket. The Hamiltonian for the Vlasov–Maxwell system is

$$\tilde{H} = mc^2\int d^3\mathbf{r} d^3\mathbf{p}\gamma f + \frac{1}{8\pi}\int d^3\mathbf{r}(|\mathbf{E}|^2 + |\mathbf{B}|^2). \tag{6}$$

To obtain equations of motion for the moments, we must express $\tilde{H}$ in terms of the moments. If we choose an explicit form for g, then by the arguments above we could write $\tilde{H}$ exactly in terms of the moments. Making use of our assumption of small momentum spread, we can expand γ about $\mathbf{P}$. Working to lowest order in Π (i.e., keeping terms through $\delta p_i \delta p_j$), knowledge of the specific form of g is not required. We find

$$H = mc^2\int d^3\mathbf{r} n\gamma_0\left(1 + \frac{\Pi_{ij}\Lambda_{ij}}{2\gamma_0^2 m^2 c^2}\right) + \frac{1}{8\pi}\int d^3\mathbf{r}(|\mathbf{E}|^2 + |\mathbf{B}|^2), \tag{7}$$

where

$$\Lambda_{ij} = \delta_{ij} - \frac{P_i P_j}{\gamma_0^2 m^2 c^2}, \tag{8}$$

and $\gamma_0 = \sqrt{1 + P^2/m^2c^2}$. It is in expanding H that the asymptotic nature of our approximation becomes clear; the next contribution to H involves terms that scale as $|\Pi|^2/(m^4c^4\gamma_0^4)$.

We can now determine the equations of motion for the moments from the bracket and Hamiltonian in the usual way: $\partial_t n = \{n, H\}_M$, $\partial_t \mathbf{P} = \{\mathbf{P}, H\}_M$ and $\partial_t \Pi_{ij} = \{\Pi_{ij}, H\}_M$. After some algebra we find

$$\partial_t n + \nabla \cdot n\mathbf{u} = 0, \tag{9a}$$

$$\partial_t \mathbf{P} + \mathbf{u} \cdot \nabla \mathbf{P} = q(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}) - \frac{1}{n} \nabla \cdot \mathsf{p}, \tag{9b}$$

$$\partial_t \Pi_{ij} + u_k \partial_k \Pi_{ij} = -\Pi_{ik} \partial_j u_k - \Pi_{jk} \partial_i u_k + \frac{\mathsf{p}_{ki}}{n} \times (\partial_k \mathfrak{p}_j - \partial_j \mathfrak{p}_k) + \frac{\mathsf{p}_{kj}}{n} (\partial_k \mathfrak{p}_i - \partial_i \mathfrak{p}_k), \tag{9c}$$

where

$$u_k = \frac{P_k}{m\gamma_0} \left[1 - \frac{\Pi_{ii}}{2\gamma_0^2 m^2 c^2} + \frac{3}{2} \frac{P_i \Pi_{ij} P_j}{\gamma_0^4 m^4 c^4} \right] - \frac{\Pi_{ki} P_i}{\gamma_0^3 m^3 c^2}, \tag{10}$$

and we have defined the pressure tensor p by

$$\mathsf{p}_{ij} = \frac{n}{\gamma_0 m} \Lambda_{ik} \Pi_{kj}, \tag{11}$$

so as to give the conventional form on the right-hand side of the momentum equation (which is required to have the correct nonrelativistic limit). Interestingly, although Π is symmetric, p is not. While the continuity and momentum equations have a familiar form, it is noteworthy that the advection velocity $\mathbf{u}$ is not simply $\mathbf{P}/(m\gamma_0)$. Moreover, $\mathbf{u}$ is not, in general, parallel to $\mathbf{P}$. This is a consequence of thermal inertia; i.e., this is a result of $(1/n) \int d^3\mathbf{p} f \mathbf{p}/\gamma = \mathbf{P}/\gamma_0 + \mathcal{O}(\Pi)$. The fields $\mathbf{E}$ and $\mathbf{B}$ are determined by Maxwell's equations from the plasma current $\mathbf{j} = qn\mathbf{u}$. The corresponding evolution equations are $\partial_t \mathbf{E} = \{\mathbf{E}, H\}_M$ and $\partial_t \mathbf{B} = \{\mathbf{B}, H\}_M$.

Using the usual definition of the entropy per particle, S, as $nS = -\int d^3\mathbf{p} f \log f$, a simple calculation leads to $S = \log n + \frac{1}{2} \log\det\mathcal{B} + \Delta_e$, where $\Delta_e = 4\pi \int ds g(s^2) \times \log g(s^2)$. From the relation between $\mathcal{B}$ and Π, we arrive at $S = \log n - \frac{1}{2} \log\det\Pi + N \log\Delta_2 + \Delta_e$, where N is the number of dimensions. It is well known that the Vlasov–Maxwell system admits Casimir invariants of the form $\int d^3\mathbf{r} d^3\mathbf{p} \mathcal{F}(f)$ where $\mathcal{F}$ is any function. It is straightforward to show (by a formal Taylor expansion of $\mathcal{F}$) that these Casimirs can all be expressed as $C = \int d^3\mathbf{r} n \mathcal{G}(S)$. Using the moment bracket, (5), we can verify that these are also Casimirs of the reduced equations: $\{\cdot, C\}_M = 0$. Thus *all* of the Vlasov–Maxwell Casimirs have counterparts in our reduced theory. Not only is the moment theory Hamiltonian, it preserves the phase-space structure of the full theory. Two particularly interesting instances of these Casimirs, $\int d^3\mathbf{r} n$ and $\int d^3\mathbf{r} nS$, result in conservation of particles and conservation of total entropy, respectively. From the entropy conservation law, we can deduce that S satisfies $\partial_t S + u_k \partial_k S = 0$. In the cold-fluid limit, the helicity $h = \int d^3\mathbf{r} \mathfrak{p} \cdot \nabla \times \mathfrak{p}$ is a Casimir. In the moment theory this is not a Casimir but evolves according to

$$\dot{h} = \{h, H\}_M = \int d^3\mathbf{r} \Omega_j \left[(u_i - v_i) \partial_j P_i - \frac{1}{n} \partial_i \mathsf{p}_{ij} \right] \tag{12}$$

where $\Omega = \nabla \times \mathfrak{p}$ and $v_i = P_i/(m\gamma_0)$.

In the usual relativistic thermodynamic treatment [8,9], the assumption of collisional dominance forces the pressure to be isotropic, and one may introduce the temperature T by $\mathsf{p}_{ij} = (nT/\gamma_0)\delta_{ij}$. Isotropy of the pressure then implies $mT\delta_{ij} = \Lambda_{ik}\Pi_{kj}$, and

$$\Pi_{ij} = mT\left(\delta_{ij} + \frac{P_i P_j}{m^2 c^2} \right). \tag{13}$$

This special form for Π is, as we will see below, not structurally stable: starting from an isotropic initial condition the evolution equations will take Π out of this form. That is, our moment equations do not correspond to local thermodynamic equilibrium as there are no collisions.

To illustrate the importance of nonequilibrium effects in a low-temperature plasma, we examine the response of an initially thermalized plasma to an intense short laser pulse. For simplicity, we consider the very under-dense case in one dimension where the plasma response can be assumed to be quasistatic [11]. As we are primarily interested in the plasma response, we assume the laser to be nonevolving (i.e., the propagation distance is a fraction of the depletion length). Thus initially, the plasma is taken to be isotropic with a temperature on the order of 10 to

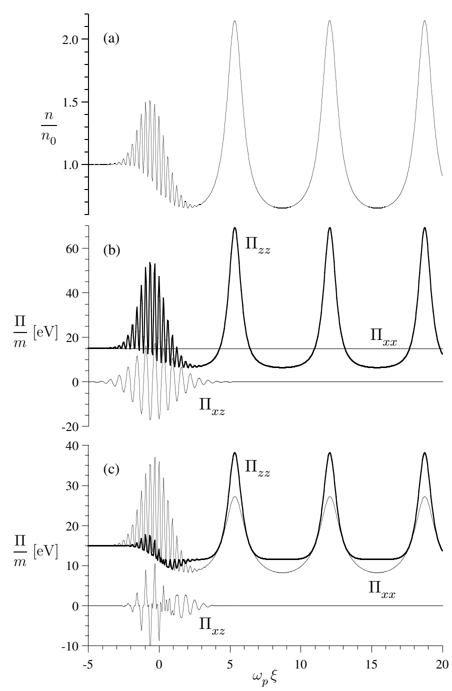

FIG. 1. Plasma response to a resonant Gaussian laser pulse with laser frequency $\omega_0 = 10\omega_p$, and dimensionless vector potential $a_0 = 1.5$: (a) density modulation; (b) $\Pi_{ij}$from (15); and (c) Π_{ij} assuming isotropic pressure.

20 eV [12–14] and negligible bulk motion. Initially, $\Pi_{ij} = mT_0\delta_{ij}$ and we see $\Pi_{ij}/(m^2c^2) \sim 10^{-5}$. Hence we can safely neglect the force due to the pressure in the momentum equation. Further, $\mathfrak{p}_x = O(\Pi)$ which allows us to drop the terms involving the pressure in the Π equations of motion. In this limit, Π is driven by the cold, quasistatic fields. For a linearly polarized laser propagating in the z direction, we transform to the co-moving coordinates $(t, z) \mapsto (t, \xi = t - z/c)$, yielding for the Π evolution equations

$$(1 - \beta_z)\partial_\xi \Pi_{xx} = 0, \tag{14a}$$

$$(1 - \beta_z)\partial_\xi \Pi_{xz} = \Pi_{xx}\partial_\xi\beta_x + \Pi_{xz}\partial_\xi\beta_z, \tag{14b}$$

$$(1 - \beta_z)\partial_\xi \Pi_{zz} = 2\Pi_{zx}\partial_\xi\beta_x + 2\Pi_{zz}\partial_\xi\beta_z, \tag{14c}$$

where $\beta = \mathbf{P}/(mc\gamma_0)$. These equations can be solved:

$$\Pi_{xx} = mT_0, \tag{15a}$$

$$\Pi_{xz} = mT_0\frac{\beta_x}{1 - \beta_z} = mT_0\frac{n}{n_0}\beta_x, \tag{15b}$$

$$\Pi_{zz} = mT_0\frac{1 + \beta_x^2}{(1 - \beta_z)^2} = mT_0\left(\frac{n}{n_0}\right)^2(1 + \beta_x^2). \tag{15c}$$

Shown in Fig. 1(a) is the plasma density wave (wake) driven by a resonant Gaussian pulse with laser frequency $\omega_0 = 10\omega_p$ (ω_p is the plasma frequency) and normalized vector potential $a_0 = 1.5$, propagating in a plasma with temperature 15 eV. The components of Π as given by (15) are shown in Fig. 1(b) and the components of Π under the assumption that the pressure is isotropic are shown in Fig. 1(c). Overall there is little "heating" of the plasma by the short pulse. Consequently, in this regime, self-trapping of electrons in the wake (leading to dark current) should not be important. Thus, provided the initial plasma temperature is sufficiently small, it should be possible to operate a laser-plasma accelerator without excessive dark current, even at large wake amplitude. Qualitatively the results agree with thermodynamic arguments for an adiabatic process: the axial "temperature" ($\propto \Pi_{zz}$) increases where the plasma is compressed and decreases where the plasma is rarified. Additionally, comparing Figs. 1(b) and 1(c), we see that the isotropic assumption gives quite different results. This is a clear indication that nonequilibrium effects are important in this physical regime.

We have developed a Hamiltonian warm, relativistic fluid theory which does not rely on assumptions of local equilibrium nor on a collisional closure. This theory respects the phase-space constraints of the Vlasov–Maxwell systems as all Casimirs of the latter are Casimirs of the warm fluid theory. A striking effect of thermal inertia is that the fluid advection velocity is not parallel to the average momentum. For the first time, the "temperature" (momentum spread) evolution in the wake of a short laser pulse has been analyzed and found to be intrinsically anisotropic. Proper characterization of the distribution function in phase space is critical to understanding the important phenomena of particle trapping and wave breaking, as well as other nonlinear processes, in intense laser-plasma interactions.

This theory can be immediately extended to multiple species which can be used to model plasmas with larger thermal spread by superimposing multiple warm fluids. Such a theory is not susceptible to the cold-beam instability [15] and appears to be a promising numerical tool for studying large momentum spread or multimodal plasmas. This formalism can also be used to develop hybrid models where, for example, only transverse velocities are integrated out in the moments. Such a model could be used to study kinetic effects in various Raman processes without resorting to a fully six-dimensional phase space. This formalism can also be applied to the Vlasov-Einstein system to study such phenomena as the velocity anisotropy observed in globular clusters.

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