Experimental Relevance of Global Properties of Time-Delayed Feedback Control

Clemens von Loewenich and Hartmut Benner
Institute for Solid State Physics, Darmstadt University of Technology, Hochschulstraße 6, D-64289 Darmstadt, Germany

Wolfram Just*
School of Mathematical Sciences, Queen Mary/University of London, Mile End Road, London E1 4NS, United Kingdom
(Received 17 May 2004; published 18 October 2004)

We show by means of theoretical considerations and electronic circuit experiments that time-delayed feedback control suffers from severe global constraints if transitions at the control boundaries are discontinuous. Subcritical behavior gives rise to small basins of attraction and thus limits the control performance. The reported properties are, on the one hand, universal since the mechanism is based on general arguments borrowed from bifurcation theory and, on the other hand, directly visible in experimental time series.

DOI: 10.1103/PhysRevLett.93.174101 PACS numbers: 05.45.Gg, 02.30.Ks, 84.30.–r

Introduction.—Control of complex and chaotic behavior has been one of the most rapidly developing topics in applied nonlinear science for more than a decade (cf. [1] and references therein). Control problems have been discussed by engineers and applied mathematicians for more than half a century. But the emphasis of noninvasive methods, i.e., methods where the control force vanishes when the target state is reached and which are sometimes called orbit control in the engineering community, has led to new concepts like time-delayed feedback techniques [2]. Such a scheme is easy to implement in experiments to stabilize time periodic states. A deeper theoretical understanding has been gained only recently (cf., e.g., [3]). So far, the control performance has been evaluated on the basis of linear stability analysis, but no systematic treatment of global properties, such as the robustness of the control performance against perturbations or the size of basins of attraction, is available in the literature. The importance of such global features has already been emphasized from the very beginning of chaos control. For instance, it has turned out by experience that limiting the size of the control force, e.g., by a simple cutoff, increases the domain of attraction of the target state. Although there exist meanwhile software packages for analyzing global features of differential-difference equations [4], such tools are of limited use since time-delayed feedback control mainly targets at systems where a proper mathematical model is not available. Thus, generic properties of the control system are of interest and such features are difficult to estimate from numerical simulations. Here, we point out a mechanism that determines basins of attraction and the control performance in a universal way. We illustrate the experimental relevance with electronic circuit experiments.

Time-delayed feedback methods are based on the measurement of a signal $s(t)$. The control force is generated from a time-delayed difference $s(t) - s(t - \tau)$. In order to keep such a scheme noninvasive, the delay time is typically chosen to be the period of the target state. In order to improve the control performance, filtering techniques in the frequency domain may be applied [5] so that the actual generation of the control force F reads

$$F(t) = K[s(t) - s(t - \tau)] + RF(t - \tau). \qquad (1)$$

For filter parameter $R = 0$ the original Pyragas scheme is recovered where the control amplitude K yields one control parameter. The additional filter parameter R improves the control performance, in particular, when systems with fast time scales are considered. The control force is used to modulate an accessible parameter of the system so that the closed loop dynamics is given by $\dot{x} = f(x(t), F(t))$. The analytical form of this equation of motion depends, of course, on the particular system. However, even on such a general level one may predict universal features of the control performance using linear stability analysis. In particular, the control domain shows a typical shape in K-R control parameter plane (cf. Fig. 1) [6].

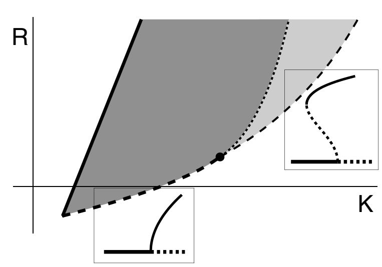

FIG. 1. Diagrammatic view of the control domain (dark and light gray shaded) for extended time-delayed feedback control. Lower control threshold (solid line); upper control threshold (broken line); saddle node bifurcation of the delay-induced orbit (dotted line). The dot indicates a transition from supercritical to subcritical behavior (cf. the insets for sketches of the bifurcation diagram). The corresponding region of bistability between the controlled orbit and the delay-induced motion is light gray shaded.

Fixing R, there exists a lower and an upper control threshold such that control is successful for control amplitudes between these critical values. The lower threshold is usually caused by a period doubling bifurcation, whereas the upper threshold involves a Hopf instability. While the left-hand boundary of the control domain is always a straight line, the precise shape of the right-hand boundary may depend on the details of the system. The general topological features are, however, confirmed by a variety of dynamical systems (cf. e.g., [7–9] for further details).

Analysis of the control performance by linear stability analysis does not take global properties such as basins of attraction into account. Such basins are of utmost importance in experimental realization since their size determines the accessibility of the target state. Depending on the type of instability that shows up at the control boundaries, a mechanism is in place that determines the basin of attraction in a universal way [10]. If the instability is supercritical then, leaving the control domain, a stable limit cycle or quasiperiodic orbit is generated by the control loop. But for subcritical instabilities an already existing unstable limit cycle collides with the stabilized orbit (cf. the insets in Fig. 1). Within the control domain this unstable object (to be precise, its stable manifold) gives rise to a finite basin of attraction. Furthermore, bistability and hysteresis are observed. In particular, the basin of attraction becomes small when the control boundary is approached. Thus subcritical behavior is an indicator for small basins of attraction and the character of the instability is crucial for the global properties of the control system.

Theoretical considerations.—Within a theoretical approach the type of instability is determined by the nonlinear contribution of the normal form equation

$$\dot{z} = \mu z - r|z|^2 z. \qquad (2)$$

The sign of the cubic coefficient r determines the type of instability. The reduced Eq. (2) for a slow variable z may be obtained from the full equation of motion by standard schemes, which can be applied even for time-delay systems [11]. The normal form reduction can be performed explicitly for a general system subjected to time-delayed feedback control (details of the formal perturbation expansion are published elsewhere [12]). The whole analysis yields two main results. First, the cubic coefficient is constant along the lower control threshold; i.e., it does not depend on R on the left-hand boundary of the control domain. In almost all cases we have observed a supercritical transition at this threshold. Second, r may change its sign along the right-hand boundary; i.e., a transition from supercritical to subcritical behavior is possible at the upper control threshold, as indicated in Fig. 1. Thus the upper control threshold may suffer from subcritical transitions which cause bistability and hysteresis and which are responsible for small basins of attraction within the control domain. Summarizing, it is subcritical behavior at the upper control threshold that causes global constraints for time-delayed feedback control.

Experimental setup.—We demonstrate the relevance of this mechanism by an electronic circuit experiment. A simple nonautonomous system is the nonlinear diode resonator sketched in Fig. 2. The circuit, which consisted of an inductor (470 μH), a resistor (51 Ω), and three parallel diodes (1N4006) acting together as a nonlinear capacitor, was sinusoidally driven at fixed frequency (340 kHz), $U(t) = U_a \sin(2\pi\nu t)$. Without control, the system undergoes a period doubling cascade to chaos on variation of the driving amplitude U_a. This scenario ensures for unstable periodic orbits with finite torsion so that these states are accessible to time-delayed feedback control [3]. We performed our experiments at $U_a =$ 4.5 V and stabilized the unstable period-one orbit. We measured the voltage at the resistor R and generated from this signal $s(t)$ our control force $F(t)$. The control loop employs multiple delay terms which exactly emulate the recursive form of Eq. (1). Finally the output of the control device was added to the driving voltage $U(t)$.

At a lower critical control amplitude the unstable orbit becomes stable through an inverse period doubling cascade, i.e., via an inverse flip bifurcation. The signal $s(t)$ becomes periodic. On further increase of the control amplitude we obtain an upper threshold where sidebands in the spectrum appear, indicating a Hopf bifurcation which leads to a quasiperiodic state. Thus, the scenario is in full accordance with general theoretical wisdom about time-delayed feedback control.

Bistability.—Experimentally the Hopf bifurcation shows off most clearly in the frequency spectrum of the signal $s(t)$. Inside the control domain we observe one sharp line indicating the frequency of the controlled orbit [cf. Fig. 3(a)]. On increasing K a sideband frequency together with its harmonics occurs directly at the Hopf bifurcation [cf. Fig. 3(b)]. But this change happens discontinuously. When decreasing K this spectrum is maintained for a larger range until the system finally jumps back to the controlled state. This kind of hysteresis in-

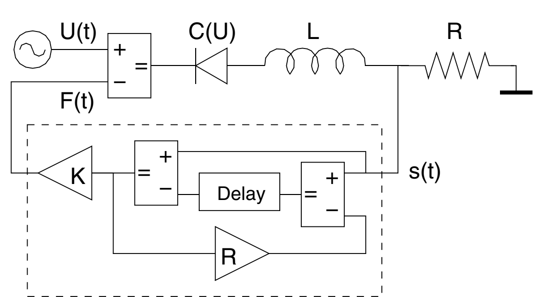

FIG. 2. Experimental setup of the nonlinear diode resonator with an extended time-delayed feedback control device.

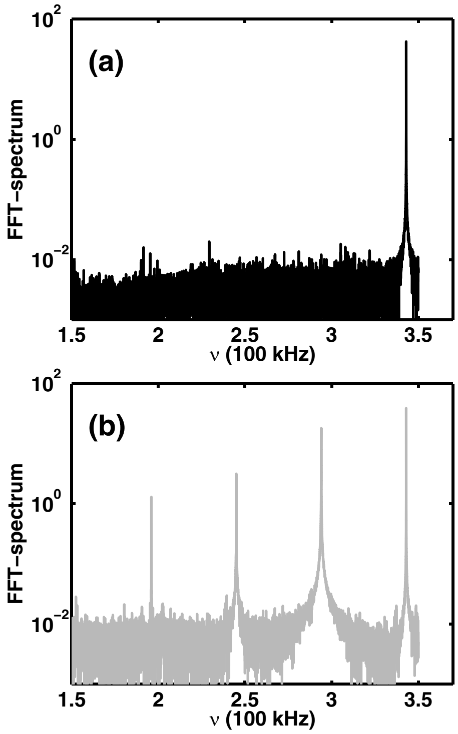

FIG. 3. Fourier spectrum of the measured signal at $K = 9.76$ and $R = 0.12$ for two different initial conditions: (a) controlled periodic orbit (adiabatic increase of the control amplitude); (b) delay-induced quasiperiodic state (adiabatic decrease of the control amplitude).

dicates that the observed Hopf bifurcation is subcritical and that a region of bistability between the controlled periodic orbit and a delay-induced quasiperiodic state occurs.

For the quantitative evaluation of the bistability we took the amplitude of the first sideband peak at about 290 kHz. Figure 4 shows the dependence on the control amplitude when K is adiabatically increased and decreased. Hysteresis and bistability is clearly visible with extremely sharp thresholds in K. At the right-hand threshold a subcritical Hopf instability takes place; i.e., a quasiperiodic peak with finite amplitude occurs in the spectrum. The left-hand threshold value, i.e., the discontinuous breakdown of the quasiperiodic state, is caused by a saddle node bifurcation (cf., e.g., [10]).

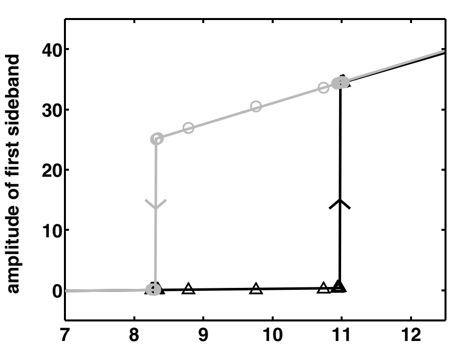

FIG. 4. Amplitude of the first sideband at 290 kHz vs control amplitude K for $R = 0.12$. Triangles, increasing K; circles, decreasing K (cf. Fig. 3 for corresponding Fourier spectra).

Since the control domain and the just mentioned threshold values strongly depend on the filter parameter R, we have probed the hysteresis for an accessible range $-0.25 < R < 0.25$. Figure 5 shows the corresponding thresholds in the K-R parameter plane. The lower threshold where control sets in and which is caused by the inverse flip bifurcation yields a straight line, in accordance with the theoretical prediction. No hysteresis was observed at this lower threshold. Thus the bifurcation is supercritical. At the upper control threshold we observe a subcritical Hopf bifurcation for all R values. The region of bistability which is bounded by the saddle node instability of the delay-induced quasiperiodic state accounts for about 30% of the whole control domain. The Hopf bifurcation at the upper boundary remained subcritical within the whole range of investigated R values, and a transition to supercritical behavior was not observed. Apart from this feature the results are in full accordance with the theoretical expectation described above (cf. Fig. 1).

Basin of attraction.—As stated previously, subcritical bifurcations pose severe constraints on the basin of attraction. We have analyzed such a property by probing the corresponding basin of attraction directly in our experiment. Our setup was modified in a way that a short pulse could be added to the driving voltage causing a deviation from the stabilized orbit. A very short but strong pulse was applied at a fixed phase of the external periodic drive. Starting from the controlled state inside the bistable regime, we observed whether the system returned back to the controlled orbit or escaped to the quasiperiodic state. We made repeated experiments by varying systematically the control parameters as well as the width and

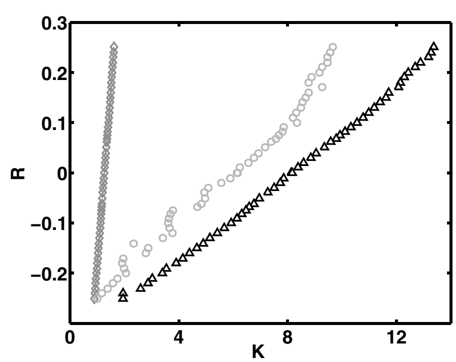

FIG. 5. Experimental results for the control thresholds in the K-R parameter plane. Diamonds, lower control threshold (supercritical flip bifurcation); triangles, upper control threshold (subcritical Hopf bifurcation); circles, collapse of the delay-induced quasiperiodic state (saddle node bifurcation).

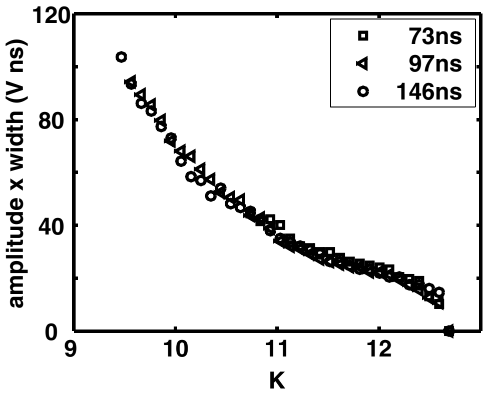

FIG. 6. Critical pulse strength in dependence on the control amplitude for a different width of pulses and $R = 0.12$.

the amplitude of the voltage pulse. As long as the strength of the pulse, i.e., the product of amplitude and width of the pulse, does not exceed a critical value we find relaxation towards the periodic orbit. This critical strength gives a measure for the size of the basin of attraction. Results are shown in Fig. 6.

First of all we find that the critical pulse strength does not depend on the precise form of the pulse. On variation of the pulse width (2.5%, 3.3%, and 5% of the period) the critical voltage amplitudes indicating the boundary of the basin of attraction scaled in the reciprocal way. Thus, we obtained a nice coincidence of our data and our experiment really probes the basin of attraction. The size of the basin may be read off from the data displayed in Fig. 6.

Second, the critical pulse strength tends towards zero when the upper control threshold is approached. That property is in full accordance with the scenario of the subcritical Hopf bifurcation since the basin of attraction becomes small as well in that limit. Furthermore, the dependence of the pulse strength on K shows an S-shape characteristic which is expected for the size of the basin according to the theoretical prediction [cf. the upper inset in Fig. 1 and the normal form analysis according to Eq. (2)]. Thus, we have striking experimental evidence that subcritical behavior is a universal mechanism which determines global features of time-delayed feedback control. Last but not least, the data displayed in Fig. 6 indicate the sensitivity of the controlled system with respect to external perturbations and thus quantify the degree of structural stability of the control scheme.

Conclusion.—For the first time we have shown, both analytically and experimentally, that the performance of time-delayed feedback control depends on whether the transitions at the control boundaries are continuous or discontinuous. A subcritical transition at the control boundary gives rise to small basins of attraction and thus limits the control scheme considerably. The relevance of such a mechanism was demonstrated by an electronic circuit experiment where a subcritical Hopf bifurcation gives rise to a pronounced hysteresis at the upper control boundary. The related basin of attraction was directly probed in the experiment by applying voltage pulses and observing the transient response of the system. Our experimental results are in excellent agreement with theoretical predictions. Since the underlying mechanism is explained in terms of bifurcation theory our findings are universal features of time-delayed feedback control. Thus, improvements of global properties of time-delayed feedback schemes have to focus on the suppression of subcritical transitions. Analytical approaches such as used in [12] may be helpful to achieve this goal. But the thorough understanding of global features of time-delayed feedback control and of time-delay systems in general is still at its infancy, and the present investigation is just a first step on a longer way.

Support by the Deutsche Forschungsgemeinschaft through Grants No. JU 261/3-1 and No. BE 864/4-1 is gratefully acknowledged.

*Electronic address: w.just@qmul.ac.uk

[1] *Handbook of Chaos Control*, edited by H. G. Schuster (Wiley-VCH, Berlin, 1999).
[2] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[3] W. Just, T. Bernard, M. Ostheimer, E. Reibold, and H. Benner, Phys. Rev. Lett. **78**, 203 (1997).
[4] K. Engelborghs, T. Luzyanina, and D. Roose, J. Comput. Appl. Math. **125**, 265 (2000).
[5] J. E. S. Socolar, D.W. Sukow, and D. J. Gauthier, Phys. Rev. E **50**, 3245 (1994).
[6] W. Just, E. Reibold, H. Benner, K. Kacperski, F. Fronczak, and J. Hołyst, Phys. Lett. A **254**, 158 (1999).
[7] J. E. S. Socolar and D. J. Gauthier, Phys. Rev. E **57**, 6589 (1998).
[8] W. Just, E. Reibold, K. Kacperski, P. Fronczak, J. Hołyst, and H. Benner, Phys. Rev. E **61**, 5045 (2000).
[9] O. Beck, A. Amann, E. Schöll, J. E. S. Socolar, and W. Just, Phys. Rev. E **66**, 016213 (2002).
[10] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1986).
[11] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations* (Springer, New York, 1993).
[12] W. Just, H. Benner, and C. v. Loewenich, Physica D (Amsterdam) (to be published).