Rotating Black Holes in Higher Dimensions with a Cosmological Constant

G. W. Gibbons,[1] H. Lü,[2,*] Don N. Page,[2,3,†] and C. N. Pope[2,*]

[1]DAMTP, Centre for Mathematical Sciences, Cambridge University,Wilberforce Road, Cambridge CB3 OWA, United Kingdom
[2]George P. & Cynthia W. Mitchell Institute for Fundamental Physics, Texas A & M University, College Station, TX 77843-4242, USA
[3]Theoretical Physics Institute, 412 Physics Lab., University of Alberta, Edmonton, Alberta T6G 2J1, Canada[‡]

(Received 21 May 2004; published 22 October 2004)

We present the metric for a rotating black hole with a cosmological constant and with arbitrary angular momenta in all higher dimensions. The metric is given in both Kerr-Schild and the Boyer-Lindquist form. In the Euclidean-signature case, we also obtain smooth compact Einstein spaces on associated S^{D-2} bundles over S^2, infinitely many for each odd $D \geq 5$. Applications to string theory and M-theory are indicated.

DOI: 10.1103/PhysRevLett.93.171102 PACS numbers: 04.20.Jb, 04.50.+h

In recent years there has been a strong interest, in both physics and mathematics, in higher-dimensional solutions of Einstein's equations. Black holes are among the most important exact solutions in general relativity, and so solutions describing higher-dimensional black holes are of particular significance. The first general rotating black hole solutions in higher dimensions were given by Myers and Perry [1], in the case that the cosmological constant vanishes. These have since been used extensively in string and M-theory calculations. More recently, interest has grown in Einstein metrics with a cosmological constant, both cosmologically in four dimensions, and in fundamental theories of nature in higher dimensions. In fact, in four dimensions, Carter [2] had already found a generalization of the Kerr solution with a cosmological constant and asymptotically de Sitter or anti–de Sitter (AdS) boundary conditions (the Kerr–de Sitter metric). Hawking, Hunter, and Taylor-Robinson [3] generalized Carter's solution to five dimensions with arbitrary angular momenta and to all dimensions with just one nonzero angular momentum parameter.

In a recent development, Tasinato *et al.* have shown that the Kerr solutions in five dimensions or higher (with zero cosmological constant) may be interpreted as time-dependent S-brane solutions of string or M-theory [4]. (See also related work on twisted S-branes and their relation to Kerr solutions in four dimensions [5] and in higher dimensions [6].) An important question is how the results of this work on time-dependent cosmological backgrounds in string and M-theory are affected by a nonvanishing cosmological term. This requires explicit solutions generalizing the higher-dimensional Kerr solution to the case when the cosmological constant is nonzero.

Another area of string and M-theory where solutions with nonvanishing cosmological constant are needed is in the AdS/CFT (conformal field theory) correspondence. Following the pioneering work of [3] in five dimensions, the principal remaining cases of interest are in dimensions six and seven. An important application of our new metrics is to study the thermodynamics of rotating black holes in higher-dimensional antide Sitter backgrounds, especially those of relevance for the AdS/CFT correspondence. Recent work in [7], using our new metrics, has provided complete results for the masses and other thermodynamic quantities in all dimensions. This has clarified certain inconsistencies in previous literature, and in fact, having the results available in all dimensions has also helped to settle some previous residual inconsistencies in four dimensions.

A further striking application of the Kerr–de Sitter metrics is in the Euclidean-signature regime, where in four dimensions they provided, by analytic continuation, the first nonsingular and compact inhomogeneous Einstein metrics with positive-definite signature and positive scalar curvature [8]. One application of this metric is as an instanton mediating creation of the universe "from nothing" [9]. This has been generalized in [10] to five dimensions, producing an infinite family of nonsingular Einstein metrics which, for example, using the AdS/CFT correspondence, provide infinitely many supersymmetry-breaking ground states for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. In a recent development, it has been shown that in limiting cases, our new Kerr–de Sitter metrics in the Euclidean regime give rise to infinite families of Einstein-Sasaki metrics [11], which can provide supersymmetric backgrounds of importance for the AdS/CFT correspondence.

Motivated by these considerations, we present here our basic results for rotating black hole metrics in all higher dimensions with a cosmological constant and with arbitrary angular momenta. A more detailed treatment is given in [12].

Let the dimension of spacetime be $D = 2N + \epsilon + 1 \geq 4$, with $N = [(D-1)/2]$ being the number of orthogonal spatial 2-planes, each of which can have a rotation parameter a_i. Thus $\epsilon = (D-1) \bmod 2$. Let ϕ_i be the N azimuthal angles in the N orthogonal 2-planes, each with period 2π. Let the remaining $N + \epsilon$ spatial dimensions be parameterized by a radial coordinate r and by $N + \epsilon$ "direction cosines" μ_i obeying the constraint

$$\sum_{i=1}^{N+\epsilon} \mu_i^2 = 1, \tag{1}$$

where $0 \le \mu_i \le 1$ for $1 \le i \le N$, and (for even D) $-1 \le \mu_{N+1} \le 1$.

The Kerr–de Sitter metrics we have found satisfy $R_{\mu\nu} = (D-1)\lambda g_{\mu\nu}$, and are given in Kerr-Schild form [13] by

$$ds^2 = d\bar{s}^2 + \frac{2M}{U}(k_\mu dx^\mu)^2, \tag{2}$$

where the de Sitter metric $d\bar{s}^2$, the null 1-form k_μ, and the function $U(r, \mu_i)$ are given by

$$\begin{aligned} d\bar{s}^2 = &-W(1-\lambda r^2)dt^2 + F dr^2 + \sum_{i=1}^{N+\epsilon} \frac{r^2+a_i^2}{1+\lambda a_i^2} d\mu_i^2 \\ &+ \sum_{i=1}^{N} \frac{r^2+a_i^2}{1+\lambda a_i^2} \mu_i^2 d\phi_i^2 + \frac{\lambda}{W(1-\lambda r^2)} \\ &\times \left(\sum_{i=1}^{N+\epsilon} \frac{(r^2+a_i^2)\mu_i d\mu_i}{1+\lambda a_i^2} \right)^2, \end{aligned} \tag{3}$$

$$k_\mu dx^\mu = F dr + W dt - \sum_{i=1}^{N} \frac{a_i \mu_i^2}{1+\lambda a_i^2} d\phi_i, \tag{4}$$

$$U = r^\epsilon \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{r^2+a_i^2} \prod_{j=1}^{N} (r^2+a_j^2), \tag{5}$$

where the functions $W(\mu_i)$ and $F(r, \mu_i)$ are defined to be

$$W \equiv \sum_{i=1}^{N+\epsilon} \frac{\mu_i^2}{1+\lambda a_i^2}, \qquad F \equiv \frac{1}{1-\lambda r^2} \sum_{i=1}^{N+\epsilon} \frac{r^2 \mu_i^2}{r^2+a_i^2}. \tag{6}$$

We have been led to these metrics by putting the previously known $D = 4$ and $D = 5$ Kerr–de Sitter metrics into Kerr-Schild form, and making natural generalizations to higher dimensions. We have explicitly checked that they obey the Einstein equation for all physically interesting cases $D \le 11$. Since $D \le 11$ is not distinguished in any way in the general expressions for the metrics, we are confident that they are valid in all dimensions. Furthermore, if all rotations a_i except one are set to zero, our expressions reduce to those obtained in [3] in any dimension. Finally, we note that if the cosmological constant is set to zero, our metrics reduce to those found by Kerr [14] and by Myers and Perry [1].

One may eliminate cross terms with dr by passing to generalized Boyer-Lindquist coordinates

$$\begin{aligned} dt &= d\tau + \frac{2M dr}{(1-\lambda r^2)(V-2M)}, \\ d\phi_i &= d\varphi_i + \frac{2M a_i dr}{(r^2+a_i^2)(V-2M)}. \end{aligned} \tag{7}$$

The Kerr–de Sitter metrics then have the form

$$\begin{aligned} ds^2 = &-W(1-\lambda r^2)d\tau^2 + \frac{2M}{U}\left(W d\tau - \sum_{i=1}^{N} \frac{a_i \mu_i^2 d\varphi_i}{1+\lambda a_i^2} \right)^2 \\ &+ \sum_{i=1}^{N} \frac{r^2+a_i^2}{1+\lambda a_i^2} \mu_i^2 d\varphi_i^2 + \frac{U dr^2}{V-2M} \\ &+ \sum_{i=1}^{N+\epsilon} \frac{r^2+a_i^2}{1+\lambda a_i^2} d\mu_i^2 + \frac{\lambda}{W(1-\lambda r^2)} \\ &\times \left(\sum_{i=1}^{N+\epsilon} \frac{r^2+a_i^2}{1+\lambda a_i^2} \mu_i d\mu_i \right)^2, \end{aligned} \tag{8}$$

where $V(r)$ is defined by

$$V \equiv \frac{U}{F} \equiv r^\epsilon - 2(1-\lambda r^2) \prod_{i=1}^{N} (r^2+a_i^2). \tag{9}$$

The Kerr–de Sitter metrics have Killing horizons at $r = r_H$, where $V(r_H) = 2M$ and where the Killing vector field

$$\begin{aligned} l &= \frac{\partial}{\partial t} + \sum_{i=1}^{N} \frac{a_i(1-\lambda r_H^2)}{r_H^2+a_i^2} \frac{\partial}{\partial \phi_i} \\ &= \frac{\partial}{\partial \tau} + \sum_{i=1}^{N} \frac{a_i(1-\lambda r_H^2)}{r_H^2+a_i^2} \frac{\partial}{\partial \varphi_i} \end{aligned} \tag{10}$$

coincides with the null generator of the horizon. The Kerr-Schild coordinates extend through the future horizon. By contrast, the Boyer-Lindquist coordinates are valid either outside the horizon or inside the horizon. It is the latter case, in which r plays the role of the time coordinate, that is relevant for time-dependent S-brane solutions.

On the horizon, the Killing vector l obeys $l^\mu \nabla_\mu l_\nu = \kappa l_\nu$, where the surface gravity, constant on each connected component of the horizon, is given by

$$\kappa = r_H(1-\lambda r_H^2)\left(\sum_{i=1}^{N} \frac{1}{r_H^2+a_i^2} + \frac{\epsilon}{2r_H^2} \right) - \frac{1}{r_H}. \tag{11}$$

The area of the horizon is given by

$$A_H = \mathcal{A}_{D-2} r_H^\epsilon - 1 \prod_{i=1}^{N} \frac{r_H^2+a_i^2}{1+\lambda a_i^2}, \tag{12}$$

where

$$\mathcal{A}_m = \frac{2\pi^{(m+1)/2}}{\Gamma[(m+1)/2]} \tag{13}$$

is the volume of the unit m sphere.

We can pass from the Lorentzian-signature Kerr–de Sitter metrics (8) to Euclidean-signature Einstein metrics by making the Boyer-Lindquist time coordinate τ and the rotation parameters a_i all purely imaginary. The generic local Einstein metrics do not give smooth complete compact Einstein spaces, but for $\lambda > 0$, we can choose discrete special values for a_i and M, as in four dimensions in

[8] and in five dimensions in [10], to get complete nonsingular metrics.

The idea is that $i\tau$ becomes an angular coordinate with the appropriate period required to avoid a conical singularity at one root of $V(r) - 2M$, say at $r = r_1$. We call this the black hole horizon, by analogy with the Lorentzian-signature case. If r ranges from r_1 to a second root of $V(r) - 2M$, say at $r = r_2$ (which we shall call the cosmological horizon), we require the same period of $i\tau$ to avoid a conical singularity at $r = r_2$. Thus the surface gravities at r_1 and at r_2 must be identical, which can be accomplished by choosing M so that r_1 approaches r_2. In this limit g_{rr} diverges in just the right way that the proper distance between the two roots or horizons approaches a nonzero finite limit. In the limiting process, the period of $i\tau$ goes to infinity, but the metric length of its orbit remains finite. After appropriately rescaling r and $i\tau$, one arrives at a finite metric.

The remaining conditions for regularity are that in each 2-plane with a nonzero rotation parameter a_i, the black hole horizon rotates an integer number k_i times, relative to the cosmological horizon, during one period of the Euclidean time coordinate $i\tau$. More details are given in [12]. These conditions place N constraints on the N rotation parameters a_i.

One obtains [12] smooth compact Einstein metrics of the form

$$\begin{aligned}\lambda ds^2 = {} & \frac{(1+A)z(\mu_i)}{4A+2A^2+2B}(d\chi^2 + \sin^2\chi d\psi^2) \\ & + \sum_{i=1}^{N+\epsilon}\frac{(1+A)d\mu_i^2}{1+A+x_i} + \frac{A}{w(\mu_i)}\left(\sum_{i=1}^{N+\epsilon}\frac{(1+A)\mu_i d\mu_i}{1+A+x_i}\right)^2 \\ & + \sum_{i=1}^{N}\frac{(1+A)\mu_i^2}{1+A+x_i}\left(d\varphi_i + k_i \sin^2\frac{\chi}{2}d\psi\right)^2 \\ & - \frac{1+A}{z(\mu_i)}\left[\sum_{i=1}^{N}\frac{\sqrt{x_i+x_i^2}\,\mu_i^2}{1+A+x_i}\left(d\varphi_i + k_i\sin^2\frac{\chi}{2}d\psi\right)\right]^2,\end{aligned} \tag{14}$$

where

$$A \equiv \sum_{i=1}^{N} x_i - \frac{\epsilon}{2}, \qquad B \equiv \sum_{i=1}^{N} x_i^2 + \frac{\epsilon}{2},$$

$$w(\mu_i) \equiv \sum_{i=1}^{N+\epsilon}\frac{Ax_i\mu_i^2}{1+A+x_i}, \qquad z(\mu_i) \equiv \sum_{i=1}^{N+\epsilon} x_i\mu_i^2, \tag{15}$$

and where the parameters x_i must be chosen so that each

$$k_i = \frac{2(1+A+x_i)\sqrt{x_i+x_i^2}}{2A+A^2+B} \tag{16}$$

is an integer.

In terms of the constant parameters x_i defined here and the resulting auxiliary constants A and B, the roots of $V(r) - 2M$ and the parameters of the Kerr–de Sitter metric (8) are given by

$$r_1 = r_2 = \sqrt{\frac{1+A}{\lambda A}}, \qquad M = -\frac{r_1^{D-3}}{2A}\prod_{i=1}^{N}(-x_i^{-1}), \qquad a_i = \mathrm{i} r_1\sqrt{\frac{1+x_i}{x_i}}. \tag{17}$$

One can see that for a real Euclidean metric, one needs either all $x_i \le -1$ or all $x_i > 0$. In the former case, we showed [12] that all but one x_i must be -1, corresponding to only one a_i and k_i nonzero, and that the only nonzero k_i allowed is $k_i = 1$. The resulting solutions were first given in [8] for $D = 4$ and in [10] for higher D. In the latter case, where all x_i are positive, which is allowed only for odd D, we showed [12] that all possible sets of purely positive integers k_i lead to unique solutions for $x_i > 0$ and to unique regular compact Einstein metrics, though $k_1 = k_2 = 1$ for $D = 5$ leads to $x_1 = x_2 = \infty$ rather than to finite solutions for the x_i. This and certain other cases in which one or more $x_i = \infty$ also lead to regular metrics [10].

Except for the $D = 5$ solutions with $k_1 \ge 1$ and $k_2 \ge 1$, which were given by [10], our compact Einstein metrics with all $k_i > 0$ in odd D are new. Because all sets of positive k_i are allowed for all odd $D \ge 5$ we get an infinite set of smooth compact Einstein metrics on $S^2 \times S^{D-2}$ when $\sum_i k_i$ is even, and an infinite set of smooth compact Einstein metrics on the nontrivial S^{D-2} bundle over S^2 when $\sum_i k_i$ is odd.

We thank Tekin Dereli, Sean Hartnoll, and Yukinori Yasui for helpful discussions. G.W.G. and D.N.P. are grateful to the George P. & Cynthia W. Mitchell Institute for Fundamental Physics for hospitality during the course of this work.

*Research supported in part by DOE Grant No. DE-FG03-95ER40917.
†Research supported in part by the Natural Science and Engineering Research Council of Canada.
‡Permanent address for D. N. P.

[1] R. C. Myers and M. J. Perry, Ann. Phys. (Berlin) **172**, 304 (1986).
[2] B. Carter, Commun. Math. Phys. **10**, 280 (1968); *Black Hole Equilibrium States*, in Black Holes (Les Houches Lectures), edited B. S. DeWitt and C. DeWitt (Gordon and Breach, New York, 1972).
[3] S.W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, Phys. Rev. D **59**, 064005 (1999).
[4] G. Tasinato, I. Zavala, C. P. Burgess, and F. Quevedo, J. High Energy Phys. 04 (2004) 038.
[5] J. E. Wang, J. High Energy Phys. 04 (2004) 066.
[6] H. Lü and J. F. Vazquez-Poritz, J. High Energy Phys. 04 (2004) 050.

[7] G.W. Gibbons, M..J Perry, and C. N. Pope, hep-th/0408217.
[8] D. N. Page, Phys. Lett. **B79**, 235 (1978); reprinted in *Euclidean Quantum Gravity*, edited by G.W. Gibbons and S.W. Hawking (World Scientific, Singapore, 1993), where the first factor in the denominator of the penultimate line of Eq. (41) is corrected from $(3 - \nu^2)^2$ to $4(3 + \nu^2)^2$.
[9] G.W. Gibbons and H. J. Pohle, Nucl. Phys. **B410**, 117 (1993).
[10] Y. Hashimoto, M. Sakaguchi, and Y. Yasui, hep-th/0402199.
[11] Y. Hashimoto, M. Sakaguchi, and Y. Yasui, hep-th/0407114.
[12] G.W. Gibbons, H. Lü, D. N. Page, and C. N. Pope, hep-th/0404008 [J. Geom. Phys. (to be published)].
[13] R. P. Kerr and A. Schild, Proc. Symp. Appl. Math. **17**, 199 (1965).
[14] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).