Spin Chains and String Theory

Martin Kruczenski
Department of Physics, Brandeis University Waltham, Massachusetts 02454, USA
(Received 12 April 2004; published 15 October 2004)

Recently, an important test of the anti de Sitter/conformal field theory correspondence has been done using rotating strings with two angular momenta. We show that such a test can be described more generally as the agreement between two actions: one a low energy description of a spin chain appearing in the field theory side, and the other a limit of the string action in $AdS_5 \times S^5$. This gives a map between the mean value of the spin in the boundary theory and the position of the string in the bulk, and shows how a string action can emerge from a gauge theory in the large-N limit.

DOI: 10.1103/PhysRevLett.93.161602 PACS numbers: 11.25.Tq, 11.15.–q, 75.10.Pq

A few years ago, Maldacena [1] proposed an equivalence between string theory propagating in anti de Sitter (AdS) space and a particular conformal field theory (CFT): a gauge theory known as $\mathcal{N} = 4$ super Yang-Mills theory ($\mathcal{N} = 4$ SYM for short). Therefore this correspondence, usually known as the AdS/CFT correspondence, promised to reveal the deep relation between gauge theory and string theory that was conjectured to exist in 't Hooft's large-N limit [2]. In a recent paper [3], Berenstein, Maldacena, and Nastase (BMN) made the first steps in that direction by showing that certain operators in the boundary theory corresponded to string excitations in the bulk. It was later observed that such relation followed from a more general relation between semiclassical rotating strings in the bulk [4] and certain operators in the boundary. A lot of activity followed those papers and many new rotating solutions were found [5]. In a parallel development, Minahan and Zarembo [6] observed that the one-loop anomalous dimension of operators composed of scalars in $\mathcal{N} = 4$ SYM theory follows from solving an integrable spin chain [25]. Subsequently [8], much attention was devoted to the subset of operators given by

$$\mathcal{O}^{J_1,J_2} = \mathrm{Tr} ZZXZXX \ldots ZX, \tag{1}$$

where the right-hand side contains and arbitrary permutation containing a number J_1 of X's and J_2 of Z's. Here we denote $X = \Phi^1 + i\Phi^2, Z = \Phi^3 + i\Phi^4$, and $\Phi^a, a = 1\ldots 6$ are the adjoint scalars of $\mathcal{N} = 4$. There are as many such operators as different permutations of the X and Z one can make up to cyclic permutations. In the free theory all these operators have conformal dimension $\Delta_0 = J_1 + J_2 = J$. The one-loop anomalous dimension can be obtained from the 1-loop dilatation operator which, acting on these operators, takes the form (in the large-N limit) [9]:

$$D_{1-\mathrm{loop}} = \tilde{\lambda} \sum_{i=1}^{J} \left(\frac{1}{4} - \vec{S}_i \vec{S}_{i+1} \right), \qquad \text{with} \quad \tilde{\lambda} = \frac{\lambda}{4\pi^2} = \frac{g_{YM}^2 N}{4\pi^2}. \tag{2}$$

To apply $D_{1-\mathrm{loop}}$ to $\mathcal{O}^{J_1,J_2}$, one should consider $\mathcal{O}^{J_1,J_2}$ as a spin $1/2$ chain identifying, e.g., X with a spin down state $|\downarrow\rangle$ and Z with a spin up $|\uparrow\rangle$:

$$ZZXZXX \ldots ZX \Longleftrightarrow |\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow \ldots \uparrow\downarrow\rangle. \tag{3}$$

After this identification, the spin operators $\vec{S}$ act in the usual manner. The trace in (1) implies that we have to consider periodic chains ($\vec{S}_{J+1} = \vec{S}_1$) and zero momentum states, *i.e.*, invariant under cyclic permutations. Thus, the computation of one-loop anomalous dimensions reduces to the diagonalization of a spin $1/2$ ferromagnetic Heisenberg chain of length J and coupling constant $\tilde{\lambda}$. The authors of [10] used Bethe ansatz techniques as in [6,8] to obtain an operator, linear combination of the $\mathcal{O}^{J_1,J_2}$, whose one-loop anomalous dimension $\gamma = \epsilon_1(\frac{J_1}{J_2})\frac{\lambda}{J}$ precisely agreed with the first subleading term in the large J expansion of the energy of a rotating string [10] $E = J + \epsilon_1(\frac{J_1}{J_2})\frac{\lambda}{J} + \mathcal{O}(\frac{\lambda^2}{J^3})$. More precisely this is an expansion valid for $J \to \infty$ keeping $\frac{\lambda}{J^2}$ fixed and small. The fact that the nontrivial function $\epsilon_1(\frac{J_1}{J_2})$ is the same on both sides of the correspondence is the main result of [10] and was further discussed in [14] following [12,13].

Here we go one step further and show that the spin chain system, in the limit we are interested in, is described by a sigma model which precisely agrees with the sigma model obtained from the rotating string in the same limit. The identification then goes beyond a particular solution and gives a precise mapping of the states since we show that the mean value of the spin at a given site is the same as the position of the string in the bulk. That is, an operator, linear combination of the $\mathcal{O}^{J_1,J_2}$ is equivalent to a state of the spin chain that we can map into a string configuration making the identification complete.

Heisenberg chain.—The Heisenberg model (2) has been studied over the years (see, e.g., [15]) as an example of ferro and anti-ferro magnetism and as a test bed for new theoretical developments. Using coherent states it can be written in a path integral formulation [16]. In the case of spin $= \frac{1}{2}$, the coherent states, labeled by two angles ϕ and θ, are defined as $|\vec{n}\rangle = e^{iS_z\phi} e^{iS_y\theta}|\uparrow\rangle$ where $\vec{n}$ is a unit

 0031-9007/04/93(16)/161602(4)$22.50

vector with components $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and $|\uparrow\rangle$ is the state with $S_z = \frac{1}{2}$. A path integral representation for transition amplitudes can be derived with an action

$$S = \frac{1}{2}\sum_{i=1}^{J}\int dt\left[\int_0^1 d\xi \vec{n}_i\cdot(\partial_t\vec{n}_i\times\partial_\xi\vec{n}_i) - \frac{\tilde{\lambda}}{8}(\vec{n}_i - \vec{n}_{i+1})^2\right]. \tag{4}$$

The first term is a Wess-Zumino term proportional to the area spanned between the trajectory and the North Pole. Its definition requires the introduction of an additional coordinate ξ with $\theta(\xi = 1) = 0$. Since this action is linear in time derivatives, near the ground state the dispersion relation is quadratic ($\omega \sim k^2$) as is known to be for ferromagnetic magnons. In our case this is the BMN limit of operators (1) where $\gamma \sim n^2$.

Therefore, for long chains, the low energy states can be described as long wavelength spin waves. Thus, one can consider the continuum limit of (4) and use a continuous coordinate σ running from 0 to J. The action, in terms of angles, becomes

$$S = -\int dt\int_0^J d\sigma\left\{\frac{1}{2}\cos\theta\partial_t\phi + \frac{\tilde{\lambda}}{8}\left[(\partial_\sigma\theta)^2 + \sin^2\theta(\partial_\sigma\phi)^2\right]\right\}. \tag{5}$$

From here we obtain the momentum, Hamiltonian and momentum density in direction σ:

$$\mathcal{P}_\phi = S_z = -\frac{1}{2}\int_0^J \cos\theta d\sigma, \tag{6}$$

$$\mathcal{H} = \frac{\tilde{\lambda}}{8}\int_0^J d\sigma[(\partial_\sigma\theta)^2 + \sin^2\theta(\partial_\sigma\phi)^2], \tag{7}$$

$$P = \int_0^J T_{01}d\sigma = -\frac{1}{2}\int_0^J \cos\theta\partial_\sigma\phi = 0, \tag{8}$$

where the last equality is a condition on the solutions corresponding to taking operators invariant under cyclic permutations.

In the continuum limit, the action is not renormalized because the Wess-Zumino term is topological and the coupling constant is fixed by comparing small fluctuations with the exact result for spin waves with only one spin down.

To compare with [10], we should take the limit $J \to \infty$ keeping λ/J^2 fixed which can be more easily understood [17] by rescaling σ into $\bar{\sigma} = \frac{2\pi}{J}\sigma$. The action is obtained from (5) through the replacement

$$\int_0^J d\sigma \to J\int_0^{2\pi}\frac{d\bar{\sigma}}{2\pi}, \qquad \tilde{\lambda} = \frac{\lambda}{4\pi^2} \to \frac{\lambda}{J^2}. \tag{9}$$

Since λ/J^2 is kept fixed, the action can be written as $S = J\bar{S}$ where $\bar{S}$ is fixed and we take $J \to \infty$. The path integral is then dominated by the saddle points and so, in this limit we only need to consider classical solutions. As a check, we find now a particular classical solution with a given value of $S_z = (J_2 - J_1)/2$ and compare with [10] where the Bethe ansatz was used instead. The equations of motion are

$$\sin\theta\partial_t\theta + \frac{\tilde{\lambda}}{2}\partial_\sigma(\sin^2\theta\partial_\sigma\phi) = 0, \tag{10}$$

$$\sin\theta\partial_t\phi + \frac{\tilde{\lambda}}{2}\partial_\sigma^2\theta - \frac{\tilde{\lambda}}{2}\sin\theta\cos\theta(\partial_\sigma\phi)^2 = 0, \tag{11}$$

with periodic boundary conditions in $\sigma \in [0, J]$. Now we can make the ansatz $\partial_\sigma\phi = 0$ which trivially satisfies the zero momentum condition (8). Equation (10) implies that $\partial_t\theta = 0$ and (11) that $\partial_t^2\phi = 0$. So we can put $\partial_t\phi = w$ and have to solve $\partial_\sigma^2\theta + \frac{2w}{\tilde{\lambda}}\sin\theta = 0$. This integrates to $\partial_\sigma\theta = \pm\sqrt{a + b\cos\theta}$, with $b = \frac{4w}{\tilde{\lambda}}$ and a is a constant of integration. If $b > |a|$, at $\theta_0 = \arccos(-a/b)$ the square root becomes zero and we can change branches, namely, the "particle" returns oscillating between $-\theta_0 < \theta < \theta_0$. We concentrate on this solution and leave the case $a > |b|$ for the interested reader.

For that solution we compute the angular momenta and energy as:

$$S_z = -2\int_0^{\theta_0}\frac{\cos\theta d\theta}{\sqrt{a + b\cos\theta}} = -2\sqrt{\frac{2}{b}}\{2E(x) - K(x)\},$$

$$J = \int_0^J d\sigma = 4\int_0^{\theta_0}\frac{d\theta}{\sqrt{a + b\cos\theta}} = 4\sqrt{\frac{2}{b}}K(x), \tag{12}$$

$$E = \frac{\tilde{\lambda}}{2}\int_0^{\theta_0}\frac{a + b\cos\theta}{\sqrt{a + b\cos\theta}}d\theta = \frac{\tilde{\lambda}}{8}(aJ - 2bS_z),$$

with $x = \frac{a+b}{2b}$. The integrals were evaluated in terms of elliptic integrals [18] (We follow [10] which uses x rather than $\sqrt{x}$ as the argument of the elliptic integrals). Finally, using that $S_z = \frac{J_2 - J_1}{2}$, simple algebra leads to

$$\frac{J_2}{J} = 1 - \frac{E(x)}{K(x)}, \tag{13}$$

$$\gamma = E = 8\frac{\tilde{\lambda}}{J}K(x)[E(x) - (1 - x)K(x)]. \tag{14}$$

If we now replace from (2), $\tilde{\lambda} \to \lambda/4\pi^2$ we get, for $\gamma = \frac{\lambda}{J}\epsilon_1(\frac{J_1}{J_2})$, exactly the same result as in [10] (see eq. (2.7) there).

More interestingly, at a fixed instant in time, when sigma is varied, the end point of $\vec{n}$ goes from $\theta = 0$ to $\theta = \theta_0$ and back, then to the other side. Furthermore, each point is precessing around the z axis with the same angular velocity w. So the configuration looks exactly like a folded rotating string.

Rotating string.—The rotating string solutions corresponding to the operators discussed in the previous section were found in [11]. The relevant part of the metric in the coordinates used in that paper is

$$ds^2 = -dt^2 + d\Omega_{[3]}^2 = -dt^2 + d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2. \quad (15)$$

Changing coordinates to

$$\phi_1 = t + \varphi_1 + \varphi_2, \qquad \phi_2 = t + \varphi_1 - \varphi_2$$

we get a metric

$$ds^2 = 2dtd\varphi_1 + 2dt\cos(2\psi)d\varphi_2 + d\psi^2 + d\varphi_1^2 + d\varphi_2^2 + 2\cos(2\psi)d\varphi_1 d\varphi_2. \quad (16)$$

After making the gauge choice $t = \kappa\tau$, the Polyakov action describing a string in this background becomes:

$$S = \frac{R^2}{4\pi\alpha'}\int G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu - G_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu \quad (17)$$

$$= \frac{R^2}{4\pi\alpha'}\int 2\kappa[\dot\varphi_1 + \cos(2\psi)\dot\varphi_2] + \dot\psi^2 + \dot\varphi_1^2 + \dot\varphi_2^2 + 2\cos(2\psi)\dot\varphi_1\dot\varphi_2 - \psi'^2 - \varphi_1'^2 - \varphi_2'^2 - 2\cos(2\psi)\varphi_1'\varphi_2', \quad (18)$$

where we denote derivatives with respect to τ with a dot and those with respect to σ with a prime. The Virasoro constraints are:

$$G_{\mu\nu}\partial_\tau X^\mu \partial_\sigma X_\nu = 2\kappa[\varphi_1' + \cos(2\psi)\varphi_2'] + \dot\psi\psi' + \dot\varphi_1\varphi_1' + \dot\varphi_2\varphi_2' + 2\cos(2\psi)\dot\varphi_1\varphi_2' + 2\cos(2\psi)\dot\varphi_2\varphi_1' = 0, \quad (19)$$

and

$$G_{\mu\nu}\partial_\tau X^\mu \partial_\tau X^\nu + G_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu = 2\kappa[\dot\varphi_1 + \cos(2\psi)\dot\varphi_2] + \dot\psi^2 + \dot\varphi_1^2 + \dot\varphi_2^2 + 2\cos(2\psi)\dot\varphi_1\dot\varphi_2 + \psi'^2 + \varphi_1'^2 + \varphi_2'^2 + 2\cos(2\psi)\varphi_1'\varphi_2' = 0, \quad (20)$$

where we have also used the gauge choice $t = \kappa\tau$. Up to now we did not do any approximation. Comparing with the large angular momentum limit of the solution in [11], we see that the motion of the string is mainly captured by the rotation we just did through the change of coordinates. We assume then that all time derivatives are small. In particular, if all time derivatives are zero, we get a BPS state corresponding to a massless point like string moving around a circle. From the action and the constraints one sees that a nontrivial limit is obtained by taking, for all coordinates X^μ:

$$\dot X^\mu \to 0, \qquad \kappa \to \infty, \qquad \text{with } \kappa\dot X^\mu \text{ fixed.} \quad (21)$$

Since $J \sim \kappa$ this is a large angular momentum limit similar to the BMN limit [3,26], but since we take the limit in the action and not in the metric, we keep interactions such as $2\cos(2\psi)\varphi_1'\varphi_2'$ which do not appear in the BMN case.

In the limit (21), the action reduces to

$$S = \frac{R^2}{4\pi\alpha'}\int \{2\kappa[\dot\varphi_1 + \cos(2\psi)\dot\varphi_2] - \psi'^2 - \varphi_1'^2 - \varphi_2'^2 - 2\cos(2\psi)\varphi_1'\varphi_2'\}, \quad (22)$$

and the constraints become

$$2\kappa[\varphi_1' + \cos(2\psi)\varphi_2'] = 0, \quad (23)$$

$$2\kappa[\dot\varphi_1 + \cos(2\psi)\dot\varphi_2] + \psi'^2 + \varphi_1'^2 + \varphi_2'^2 + 2\cos(2\psi)\varphi_1'\varphi_2' = 0 \quad (24)$$

We can use any of them to eliminate φ_1. That we get the same result using one or the other is guaranteed by the equations of motion. The only point is that, since the string is closed, φ_1 is a periodic function of sigma implying $0 = \int d\sigma\varphi_1' = -\int d\sigma\cos(2\psi)\varphi_2'$, which, as we will see below is condition (8).

Replacing the first constraint in the action we get

$$S = \frac{R^2}{4\pi\alpha'}\int 2\kappa\dot\varphi_1 + 2\cos(2\psi)\kappa\dot\varphi_2 - \psi'^2 - \sin^2(2\psi)\varphi_2'^2, \quad (25)$$

which already looks quite similar to the sigma model we had before. To make the agreement precise we first compute the angular momentum

$$J = \mathcal{P}_{\varphi_1} = \frac{R^2}{4\pi\alpha'}2\kappa\int_0^{2\pi} d\sigma = \int_0^J d\tilde\sigma, \quad (26)$$

where we defined

$$\tilde\sigma = \frac{R^2}{4\pi\alpha'}2\kappa\sigma, \quad (27)$$

so that the length of the chain is J as before. To compare the energies we rescale τ into $t = \kappa\tau$. Also, one can see that the angles are related by $\varphi_2 = -\frac{1}{2}\phi, \psi = \frac{1}{2}\theta$. Finally if we use the AdS/CFT relation $R^2/\alpha' = \sqrt{\lambda}$ and drop the total derivative $\kappa\dot\varphi_1$, we get the action

$$S = -\int dt d\tilde\sigma\left\{\frac{1}{2}\cos\theta\partial_t\phi + \frac{\lambda}{32\pi^2}((\partial_\sigma\theta)^2 + \sin^2\theta(\partial_\sigma\phi)^2)\right\}, \quad (28)$$

which precisely agrees with (5). The identification of the angles implies that we can map directly a configuration $\vec n(\sigma)$ into a particular shape of the string in the bulk regardless if it is a solution or not. Since $\vec n$ is the average value of the spin at a site, $\langle\vec n|\vec S|\vec n\rangle = \frac{1}{2}\vec n$ one can identify the average value of the spin with the position of the corresponding portion of the string in the bulk. However, this also implies that, if we look at the string carefully enough, we will see the discrete nature of the spin chain.

The agreement between (5) and (28) is our main result. We can go a bit further if we consider the two-loop dilatation operator [9]

$$D_{2-\mathrm{loops}} = \frac{\lambda^2}{128\pi^4}\left[-\frac{3}{2}J + 8\sum_k \vec{S}_k\vec{S}_{k+1} - 2\sum_k \vec{S}_k\vec{S}_{k+2}\right]. \tag{29}$$

Replacing $\vec{S}_i \to \frac{1}{2}\vec{n}_i$ and Taylor expanding n in the continuum limit we get a sigma model which is quartic in derivatives. For slowly varying fields $\vec{n}$, since the length of the chain is J, we estimate that $\partial_\sigma \vec{n} \sim \frac{1}{J}$. For the one-loop term this gives $\gamma_1 \sim \lambda \int (\partial_\sigma \vec{n})^2 \sim \lambda J \frac{1}{J^2} \sim \frac{\lambda}{J}$ as we obtained before. At two-loops we have $\gamma_2 \sim \lambda^2 \int \partial_\sigma^2 \vec{n} \partial_\sigma^2 \vec{n} \sim \frac{\lambda^2}{J^3}$ as we expect if the BMN limit is well defined. Similarly, at higher loops, the full dilatation operator, in the long wavelength limit, should have the schematic expansion

$$D = J + S_{WZ} + \int_0^J d\sigma \sum_{p_1,\ldots,p_l} \lambda^{p_1+\cdots+p_l} C_{p_1,\ldots,p_l} \partial_\sigma^{p_1}\vec{n}\ldots\partial_\sigma^{p_l}\vec{n}. \tag{30}$$

Another situation that can be interesting to consider is finite temperature when the spin chain disorders. Since the expectation value of the spin gives the position of the string in space time, a disordered state looks in space time as a random walk which could be describing strings near or above the Hagedorn temperature.

In summary, we have used spin chains to describe a precise map between operators in a four dimensional gauge theory and string configurations. Although we were able to do this for a particular setup and a subset of operators, at least in principle, the same methods should be useful to derive string duals for other field theories even when the resulting string action does not correspond to a critical superstring. In this respect, many interesting theories do not have scalar fields in the adjoint but similar operators can be constructed using covariant derivatives and the gauge field strength.

I am very grateful to L. Urba for help in understanding the physics of the spin chain models, to A. Ryzhov for interesting me in this subject and for close collaboration on related issues, and also to A. Tseytlin for various comments on the first version of this Letter. This work was supported in part by NSF through Grants No. PHY-0331516 and No. PHY99-73935, and by DOE under Grant No. DE-FG02-92ER40706.

*Electronic address: martink@brandeis.edu

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998); O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rep. **C323**, 183 (2000).
[2] G.'t Hooft, Nucl. Phys. **B72**, 461 (1974); G.'t Hooft, Nucl. Phys. **B75**, 461 (1974).
[3] D. Berenstein, J. M. Maldacena, and H. Nastase, J. High Energy Phys. 04 (2002) 013.
[4] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Nucl. Phys. **B636**, 99 (2002).
[5] See A. A. Tseytlin, hep-th/0311139 for a review and references to the original literature.
[6] J. A. Minahan and K. Zarembo, J. High Energy Phys. 03 (2003) 013.
[7] V. M. Braun, S. E. Derkachov, and A. N. Manashov, Phys. Rev. Lett. **81**, 2020 (1998); V. M. Braun, S. E. Derkachov, G. P. Korchemsky, and A. N. Manashov, Nucl. Phys. **B553**, 355 (1999); A. V. Belitsky, Nucl. Phys. **B558**, 259 (1999); Phys. Lett. B **453**, 59 (1999).
[8] N. Beisert, J. A. Minahan, M. Staudacher, and K. Zarembo, J. High Energy Phys. 09 (2003) 010.
[9] N. Beisert, C. Kristjansen, and M. Staudacher, Nucl. Phys. **B664** (2003)131.
[10] N. Beisert, S. Frolov, M. Staudacher, and A. A. Tseytlin, J. High Energy Phys. 10 (2003) 037.
[11] S. Frolov and A. A. Tseytlin, Phys. Lett. B **570**, 96 (2003).
[12] G. Arutyunov and M. Staudacher, J. High Energy Phys. 03 (2004) 004.
[13] G. Arutyunov, J. Russo, and A. A. Tseytlin, Phys. Rev. D **69**, 086009 (2004).
[14] G. Arutyunov, S. Frolov, J. Russo, and A. A. Tseytlin, Nucl. Phys. **B671**, 3 (2003).
[15] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098, (1940); P.W. Anderson, Phys. Rev. **86**, 694, (1952); N. Read and S. Sachdev, Phys. Rev. Lett. **75**, 3509, (1995); A. Dhar and B. S. Shastry, Phys. Rev. Lett. **85**, 2813 (2000).
[16] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Reading, MA, 1991).
[17] See also M. Kruczenski, A. V. Ryzhov, and A. A. Tseytlin, hep-th/0403120.
[18] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals Series and Products* (Academic, New York, 2000), 6th ed.
[19] V. M. Braun, S. E. Derkachov, and A. N. Manashov, Phys. Rev. Lett. **81**, 2020 (1998); V. M. Braun, S. E. Derkachov, G. P. Korchemsky, and A. N. Manashov, Nucl. Phys. **B553**, 355 (1999); A. V. Belitsky, Nucl. Phys. **B558**, 259 (1999); Phys. Lett. B **453**, 59 (1999).
[20] U. H. Danielsson, A. Guijosa, and M. Kruczenski, J. High Energy Phys. 10 (2000) 020; J. Gomis and H. Ooguri, J. Math. Phys. (N.Y.) **42**, 3127 (2001).
[21] A. Gorsky, hep-th/0308182.
[22] A. Mikhailov, J. High Energy Phys. 12 (2003) 058.
[23] D. Mateos, T. Mateos, and P. K. Townsend, J. High Energy Phys. 12 (2003) 017.
[24] N. Beisert, Nucl. Phys. **B676**, 3 (2004).
[25] In QCD the relation between spin chains and anomalous dimensions had already been noted in [7].
[26] It also has some similarities with the so called wrapped or nonrelativistic limit [20]. For the present case some related ideas appeared in [21,22]