Construction of Non-Abelian Walls and their Complete Moduli Space

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We present a systematic method to construct exactly all Bogomol'nyi-Prasad-Sommerfield multiwall solutions in supersymmetric U(N_C) gauge theories in five dimensions with N_F hypermultiplets in the fundamental representation for infinite gauge coupling. The moduli space of these non-Abelian walls is found to be the complex Grassmann manifold $\frac{SU(N_F)}{SU(N_C) \times SU(N_F - N_C) \times U(1)}$ endowed with a deformed metric.

In constructing unified theories with extra dimensions [1] it is crucial to obtain topological defects and localization of massless or nearly massless modes on the defect. Walls in five-dimensional theories are the simplest of the topological defects leading to the four-dimensional world volume. In constructing topological defects, supersymmetry, (SUSY) theories are helpful, since partial preservation of SUSY automatically gives a solution of equations of motion [2]. These states are called Bogomol'nyi-Prasad-Sommerfield (BPS) states. The minimum number of supercharges in five dimensions is eight. Wall solutions in the U(1) gauge theory, which can be called *Abelian walls*, have been discussed with eight SUSY [3–7]. Walls in non-Abelian gauge theories have been considered in a special circumstance recently [8]. Non-Abelian solitons admitting ample moduli space structure were discovered already except for walls, and beautiful methods are available for the construction of instantons, monopoles, and vortices [9]. In this Letter, we give a systematic method to construct walls in non-Abelian gauge theories, called *non-Abelian walls*, with the gauge group U(N_C) and $N_F(>N_C)$ copies of hypermultiplets in the fundamental representation. We also find the complete moduli space for non-Abelian walls, which fills the last gap in soliton moduli spaces in the gauge-Higgs system.

We shall denote the gauge group by the uppercase suffix C, and the flavor group by F. The U(N_C) vector multiplets contain gauge fields W_M, and a real scalar field Σ, which are in the adjoint representation of U(N_C). We use an $N_C \times N_C$ matrix notation for these component fields, like $\Sigma = \Sigma^I T_I$. Here we have denoted the Hermitian generators in the Lie algebra by $T^I (I = 0, 1, 2, \cdots, N_C^2 - 1)$, satisfying the normalization condition; $\mathrm{Tr}(T_I T_J) = \frac{1}{2}\delta_{IJ}$, where T_0 is the generator of the factor U(1) gauge group. The U(1) part of vector multiplets allows the Fayet-Illiopoulos (FI) term, which gives rise to discrete vacua once mass terms for hypermultiplets are introduced. Dynamical bosons of hypermultiplets are the SU(2)$_R$ doublet of complex scalar quark fields H^{irA}. We denote space-time indices by $M, N, \cdots = 0, 1, 2, 3, 4$, and SU(2)$_R$ doublet indices by i. The color indices $r, s, \cdots$ run over $1, 2, \cdots, N_C$, whereas $A, B, \cdots = 1, 2, \cdots, N_F$ stand

for flavor indices. It is convenient to combine the N_F hypermultiplets in the fundamental representation into an $N_C \times N_F$ matrix H^i with components $(H^i)^{rA} \equiv H^{irA}$.

We shall consider a model with minimal kinetic terms for vector and hypermultiplets. The eight SUSY allow only a few parameters in our model: the masses of the Ath hypermultiplet m_A, the SU(2)$_R$ triplet of FI parameters $c_a (a = 1, 2, 3)$ for the U(1) vector multiplet, and a gauge coupling constant g for the U(N_C) gauge group. Different gauge couplings for U(1) and SU(N_C) factors can easily be incorporated, but the difference becomes irrelevant for infinite gauge coupling which we will be most interested in. After eliminating the auxiliary fields, the bosonic part of our Lagrangian reads

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}[F_{MN}(W)F^{MN}(W)] + \frac{1}{g^2}\mathrm{Tr}(\mathcal{D}^M \Sigma \mathcal{D}_M \Sigma)$$
$$+ (\mathcal{D}_M H_{irA})^\dagger \mathcal{D}^M H^{irA} - V, \tag{1}$$

where the scalar potential V is given by

$$V = \frac{g^2}{4}\mathrm{Tr}[((\sigma_a)^j{}_i H^i H_j^\dagger - c_a \mathbf{1}_{N_C})^2]$$
$$+ H_{irA}^\dagger[(\Sigma - m_A)^2]^r{}_s H^{isA}. \tag{2}$$

Here a sum over repeated indices is implied, covariant derivatives are defined by $\mathcal{D}_M H^{irA} = [\partial_M \delta_s^r + iW_M^I(T_I)^r{}_s]H^{isA}$, $\mathcal{D}_M\Sigma = \partial_M\Sigma + i[W_M, \Sigma]$, and the gauge field strength is defined by $F_{MN}(W) = -i[\mathcal{D}_M, \mathcal{D}_N]$. Our convention of metric is $\eta_{MN} = \mathrm{diag}(+1, -1, -1, -1, -1)$. In this Letter, we assume nondegenerate mass parameters m_A, which we arrange $m_A > m_{A+1}$ for all A. Our results should be valid for the degenerate mass case also, except for subtleties associated with global symmetry. Since U(1)$_F$ corresponding to a common phase is gauged, the flavor symmetry reduces to U(1)$_F^{N_F - 1}$. The SU(2)$_R$ symmetry allows us to choose the FI parameters to lie in the third direction without loss of generality as $c_a = (0, 0, c)$ with $c > 0$.

Let us discuss the vacuum structure of this model. Since we assume nondegenerate masses for hypermultiplets, we find that only one flavor $A = A_r$ ($A_r \neq A_s$, for $r \neq s$) can be nonvanishing for each color component r of hypermultiplet scalars H^{irA} with

$$H^{1rA} = \sqrt{c}\,\delta^{A_r}{}_A, \quad H^{2rA} = 0. \tag{3}$$

Here we used global gauge transformations to eliminate possible phase factors. This is called the color-flavor locking vacuum. The vector multiplet scalars Σ are determined as

$$\Sigma = \mathrm{diag}(m_{A_1}, m_{A_2}, \cdots, m_{A_{N_C}}). \tag{4}$$

We denote a SUSY vacuum specified by a set of non-vanishing hypermultiplet scalars with the flavor $\{A_r\}$ for each color component r as $\langle A_1 A_2 \cdots A_{N_C}\rangle$. Since global gauge transformations can exchange flavors A_i and A_j for the color component i and j, respectively, the ordering of the flavors $A_1, \cdots, A_{N_C}$ does not matter in considering only vacua: $\langle 1, 2, 3\rangle = \langle 2, 1, 3\rangle$. Thus a number of SUSY vacua is given by $N_F!/[(N_F - N_C)!N_C!] \equiv {}_{N_F}C_{N_C}$ and we usually take $A_1 < A_2 < \cdots < A_{N_C}$. Multiwalls are classified by topological sectors that are defined by giving two vacua at $y = \pm\infty$.

Let us obtain the BPS equations for domain walls interpolating between two SUSY vacua. We require for wall solutions that all fields depend only on the coordinate of the extra dimension $y \equiv x^4$. We also assume the Poincaré invariance on the four-dimensional world volume of the wall, implying $W_M = 0$ for the indices $M \neq y$. Note that W_y need not vanish. We demand that half of SUSY defined by $\gamma^4 \varepsilon^i = -i(\sigma^3)^i_j \varepsilon^j$ be conserved [7]. Requiring the SUSY transformation of fermions to vanish along the above SUSY directions, we find the following BPS equations for domain walls in the matrix notation

$$\mathcal{D}_y \Sigma = \frac{g^2}{2}(c\mathbf{1}_{N_C} - H^1 H^{1\dagger} + H^2 H^{2\dagger}), \tag{5}$$

$$0 = g^2 H^1 H^{2\dagger}, \tag{6}$$

$$\mathcal{D}_y H^1 = -\Sigma H^1 + H^1 M, \quad \mathcal{D}_y H^2 = \Sigma H^2 - H^2 M, \tag{7}$$

where we have used the $N_F \times N_F$ Hermitian mass matrix M defined by $(M)^A_B \equiv m_A \delta^A_B$.

If a wall configuration approaches a SUSY vacuum $\langle A_1 A_2 \cdots A_{N_C}\rangle$ at $y = +\infty$, and $\langle B_1 B_2 \cdots B_{N_C}\rangle$ at $y = -\infty$, the topological sector of the configuration is labeled by $\langle A_1 A_2 \cdots A_{N_C}\rangle \leftarrow \langle B_1 B_2 \cdots B_{N_C}\rangle$. By either performing the Bogomol'nyi completion of the energy density $\mathcal{E}$ or applying the BPS equations, we obtain the bound for the energy of the configuration

$$\int_{-\infty}^{+\infty} \mathcal{E}\,dy \geq c[\mathrm{Tr}(\Sigma)]^{+\infty}_{-\infty} = c\left(\sum_{k=1}^{N_C} m_{A_k} - \sum_{k=1}^{N_C} m_{B_k}\right). \tag{8}$$

BPS walls saturate the bound.

Let us construct solutions for BPS Eqs. (5)–(7). To this end, it is convenient to introduce an $N_C \times N_C$ invertible complex matrix function $S(y) \in GL(N_C, \mathbf{C})$ defined by [10]

$$\Sigma + iW_y \equiv S^{-1}\partial_y S. \tag{9}$$

Note that the above differential equation determines the matrix function S except for N_C^2 complex integration constants which cause an ambiguity for S. Without any assumption, the BPS Eqs. (6) and (7) dictate

$$H^1 = S^{-1}H_0 e^{My}, \quad H^2 = 0. \tag{10}$$

Here H_0 is an arbitrary complex constant $N_C \times N_F$ matrix which we call the "moduli matrix". We will postpone detailed proof (including $H^2 = 0$), for a subsequent paper. The remaining BPS equation (5) for the vector multiplets can be written in terms of the matrix S and the moduli matrix H_0. Equation (9) implies that the gauge transformations on the original fields Σ, W_y, H^1,

$$H^1 \rightarrow H^{1\prime} = UH^1,$$
$$\Sigma + iW_y \rightarrow \Sigma' + iW'_y = U(\Sigma + iW_y)U^\dagger + U\partial_y U^\dagger \tag{11}$$

can be obtained by a right multiplication of a unitary matrix $U^\dagger$ on S:

$$S \rightarrow S' = SU^\dagger, \quad U^\dagger U = 1 \tag{12}$$

without causing any transformations on the moduli matrix H_0. Therefore we obtain gauge invariant quantity Ω out of S defined by

$$\Omega \equiv SS^\dagger. \tag{13}$$

Together with the gauge invariant moduli matrix H_0, the BPS Eq. (5) can be rewritten in the following gauge invariant form

$$\partial_y^2 \Omega - \partial_y \Omega \Omega^{-1} \partial_y \Omega = g^2(c\Omega - H_0 e^{2My} H_0^\dagger). \tag{14}$$

With a suitable gauge choice, we obtain uniquely the $N_C \times N_C$ complex matrix S from the $N_C \times N_C$ Hermitian matrix Ω. Therefore, once a solution of Ω for Eq. (14) with a given moduli matrix H_0 is obtained, the matrix S can be determined and then, all the quantities, Σ, W_y, and H^1 are obtained by Eqs. (9) and (10). We find by explicit examples that gauge field W_y and/or Σ are nontrivial unlike Abelian walls.

Given the boundary conditions at both infinities $y = \pm\infty$, the differential Eq. (14) is expected to give a solution without further integration constants. Therefore the moduli matrix H_0 alone should describe the entire moduli space of walls. Equation (14) is, however, difficult to solve explicitly for finite gauge couplings g. We consider, therefore, the case of the infinite gauge coupling ($g^2 \rightarrow \infty$), where Eq. (14) for the gauge invariant Ω reduces to an algebraic equation, given by

$$\Omega_{g\rightarrow\infty} = (SS^\dagger)_{g\rightarrow\infty} = c^{-1}H_0 e^{2My} H_0^\dagger. \tag{15}$$

Therefore we can explicitly construct wall solutions in the infinite gauge coupling without solving the differen-

tial equation for Ω. This explicit solution shows clearly that the moduli space is fully covered by our moduli matrix H_0. In this limit our model reduces to a hyper-Kähler nonlinear sigma model (NLSM) whose target space is the cotangent bundle over the complex Grassmann manifold $T^*[\frac{SU(N_F)}{SU(N_C)\times SU(N_F-N_C)\times U(1)}]$ [11]. For this NLSM, our construction exhausts all possible BPS wall solutions. The NLSM has been known to be dual under $N_C \leftrightarrow N_F - N_C$ with N_F fixed. We find duality transformations of moduli matrix H_0 explicitly. For the non-Abelian gauge theory in Eqs. (1) and (2), it is likely that one needs to consider finite gauge couplings, especially if one is interested in quantum effects. The BPS domain walls in theories with eight SUSY were first obtained in hyper-Kähler NLSMs [3]. They have been the only known examples for eitght SUSY models until exact wall solutions at finite gauge coupling were found recently [6,7]. In [7] we have constructed exact wall solutions for finite gauge couplings in the case of $N_C = 1$ and $N_F = 3$ to find that their qualitative behavior is the same as the infinite gauge coupling cases found in [4]. We expect that the moduli space of walls at finite gauge couplings should be qualitatively the same as that at infinite gauge coupling. In the rest of this Letter we examine moduli matrix H_0 irrespective of finite or infinite gauge coupling.

From Eqs. (9) and (10), we find that the same original fields Σ, W_y, H^1 given by a set of matrix function S and constant moduli matrix H_0 are described by another set (S', H_0'), transformed by $V \in GL(N_C, \mathbf{C})$,

$$S \to S' = VS, \qquad H_0 \to H_0' = VH_0. \qquad (16)$$

We call this global "world-volume symmetry", which comes from the N_C^2 integration constants in solving (9). This transformation V defines an equivalence class among sets of matrix function S and moduli matrix H_0. We thus find the moduli space for multiwall solutions (without specifying boundary conditions) denoted by $\mathcal{M}_{N_F,N_C}$ is the complex Grassmann manifold:

$$\mathcal{M}_{N_F,N_C} = \{H_0 | H_0 \sim VH_0, V \in GL(N_C, \mathbf{C})\}$$

$$\equiv G_{N_F,N_C}$$

$$\simeq \frac{SU(N_F)}{SU(N_C) \times SU(N_F - N_C) \times U(1)}, \qquad (17)$$

whose complex dimension is given by $N_C(N_F - N_C)$. This is a *compact* (closed) set. On the other hand, for instance, scattering of two Abelian walls is described by a NLSM on a *noncompact* moduli space [4,7]. We also find similar noncompact moduli by an explicit analysis of multiple non-Abelian walls. These two facts can be consistently understood if we note that the moduli space $\mathcal{M}_{N_F,N_C}$ includes *all topological sectors* determined by the differ-

ent boundary conditions as we show in the rest of this Letter.

The moduli matrix H_0 contains complete data of walls including boundary conditions, number of walls, wall position, etc., Boundary conditions at $y = \pm\infty$ are most conveniently read by the following fixing of world-volume symmetry (16) :

$$H_0 = \sqrt{c}\begin{pmatrix} \cdots 0\; 1\; *\cdots\; * & \cdots & \cdots *\; e^{v_1}\; 0\cdots \\ & \cdots 0\; 1\; *\cdots & \cdots & \cdots *\; e^{v_2}\; 0\cdots \\ \vdots & & \vdots & \\ & & \cdots 0\; 1\; *\cdots & \cdots *\; e^{v_{N_C}}\; 0\cdots \end{pmatrix}, \qquad (18)$$

where all elements in the rth row before the A_rth flavor are eliminated, the A_rth flavor is normalized to be unity, and the last nonvanishing element e^{v_r} ($v_r \in \mathbf{C}$) in the rth row resides in the B_rth flavor. We can choose these flavors A_r, B_r to be ordered as

$$1 \le A_1 < A_2 < \cdots < A_{N_C} \le N_F, \qquad (19)$$

$$A_r \le B_r, \qquad B_r \ne B_s, \quad \text{for } r \ne s. \qquad (20)$$

When the set of flavors $\{B_r\}$ are not ordered like $\{A_r\}$ in Eq. (19), we must eliminate some more elements to remove the redundancy. This can be done in a well-defined procedure. We call the fixing (18) a "standard form". Since this fixing of the symmetry (16) is unique, any moduli matrix in the standard form has one-to-one correspondence with a point in the moduli space. If the moduli matrix happens to be $H_0^{rA} = \sqrt{c}\delta^{A_r}{}_A$, Eqs. (10) and (15) imply the vacuum $\langle A_1 A_2 \cdots A_{N_C}\rangle$:$H^{1rA}(y) = \sqrt{c}\delta^{A_r}{}_A$. Note that the solution H^1 in Eq. (10) implies the transformation of the moduli matrix, $H_0 \to H_0 e^{My_0}$, under a translation $y \to y + y_0$. Since the world-volume symmetry (16) allows us to multiply the matrix $(V)^r{}_s = e^{-m_{A_r}y_0}\delta^r{}_s$ from the left of H_0, the standard form (18) and the ordering of masses imply that the matrix $(VH_0 e^{My_0})^{rA}$ remains finite when taking the limit $y_0 \to \infty$ to give $\sqrt{c}\delta^{A_r}{}_A$. Thus the configuration reduces to the vacuum labeled by $\langle A_1 A_2 \cdots A_{N_C}\rangle$. Similarly, with another matrix $(V)^r{}_s = e^{-m_{B_r}y_0 - v_r}\delta^r{}_s$, we obtain $(VH_0 e^{My_0})^{rA} \to \sqrt{c}\delta^{B_r}{}_A$ in the limit of $y_0 \to -\infty$. Therefore the multiwall configuration described by the standard form (18) belongs to the *topological sector labeled by* $\langle A_1 A_2 \cdots A_{N_C}\rangle \leftarrow \langle B_1 B_2 \cdots B_{N_C}\rangle$.

A topological sector consists of all permutations of the vacuum labels $B_1, B_2, \cdots, B_{N_C}$ at $y = -\infty$. If the label $\langle B_1 B_2 \cdots B_{N_C}\rangle$ happens to be ordered, $B_1 < B_2 < \cdots < B_{N_C}$, then the moduli matrix H_0 covers generic points of the topological sector. Hence the real dimension of the topological sector is given by $2(\sum_{i=1}^{N_C} B_i - \sum_{i=1}^{N_C} A_i)$. Half of these moduli parameters represent wall positions and the rest are (quasi-)Nambu-Goldstone modes of internal symmetry. The topological sector with the largest

dimension is labeled by $\langle 1, 2, \cdots, N_C \rangle \leftarrow \langle N_F - N_C + 1, \cdots, N_F - 1, N_F \rangle$. If the label $\langle B_1 B_2 \cdots B_{N_C} \rangle$ is not ordered, H_0 has smaller dimensions as is described below in Eq. (20). We can understand this fact by noting that some walls are compressed into each other to become a single "compressed wall".

By the above observation, we find that the Grassmann manifold is decomposed into

$$\mathcal{M}_{N_F, N_C} = \sum_{\text{BPS}} \mathcal{M}_{N_F, N_C}^{\langle A_1 A_2 \cdots A_{N_C} \rangle \leftarrow \langle B_1 B_2 \cdots B_{N_C} \rangle}, \qquad (21)$$

where $\mathcal{M}_{N_F, N_C}^{\langle A_1 A_2 \cdots A_{N_C} \rangle \leftarrow \langle B_1 B_2 \cdots B_{N_C} \rangle}$ denotes the moduli subspace of BPS multiwall solutions for the topological sector of $\langle A_1 A_2 \cdots A_{N_C} \rangle \leftarrow \langle B_1 B_2 \cdots B_{N_C} \rangle$. Note that it also includes the vacuum states with no walls $\langle A_1 A_2 \cdots A_{N_C} \rangle \leftarrow \langle A_1 A_2 \cdots A_{N_C} \rangle$, which correspond to $N_F C_{N_C}$ points on the moduli space. Although each sector (except for vacuum states) is in general an open set, the total space is compact. We call $\mathcal{M}_{N_F, N_C}$ the "total moduli space". This fact is in interesting contrast to cases of other solitons like instantons, vortices, and monopoles, since the dimension of the total moduli spaces is infinite in the latter cases.

Effective Lagrangians on walls can be obtained by promoting the moduli parameters to fields on the world volume of walls [12]. The world-volume symmetry (16) naturally becomes a local gauge symmetry. Denoting the moduli fields by ϕ in $H_0(\phi)$, we obtain the Kähler metric on the total moduli space. By using explicit solutions for infinite gauge coupling, its Kähler potential is given by

$$K(\phi, \phi^*) = c \int dy \log[\det \Omega(\phi, \phi^*, y)]. \qquad (22)$$

The metric (22) is not symmetric under $SU(N_F)$ but admits an isometry $U(1)^{N_F - 1}$. Therefore the total moduli space is a deformed Grassmann manifold.

The total moduli space G_{N_F, N_C} is a special Lagrangian submanifold of the Higgs branch of vacua $T^* G_{N_F, N_C}$ of this theory. We anticipate that this is always true for arbitrary gauge groups with arbitrary matter contents.

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