Theory of Tokamak Equilibria with Central Current Density Reversal

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It is found that, with a model current profile, the Grad-Shafranov equation can be reduced to the Helmholtz equation, which can describe a variety of equilibrium configurations. With the eigenvalue problem solved in the toroidal coordinate system, an analytical solution to the Grad-Shafranov equation is found. It is demonstrated that current reversal equilibrium configurations exist with finite radial gradient of plasma pressure and continuous current density, and that current density reversal is accompanied by pressure gradient reversal.

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The concept of tokamak fusion reactors crucially depends on the sustainment of the toroidal electrical current. To realize steady-state operation of a tokamak fusion reactor, much effort has been made to improve the efficiency of current drive; however, up to date, it is still difficult to drive steady-state toroidal current with high efficiency in the central region of a tokamak. Recent progress in the tokamak fusion experiments [1,2] indicates that a tokamak may be operated with nearly zero toroidal current density in a finite central region. On the other hand, alternating-current operation of a tokamak has been demonstrated experimentally [3]. These experiments are helpful to solve the problem of sustaining plasma current. Therefore, understanding these experiments is currently one of the key issues in tokamak fusion community, and it is also of interest for those who concern the final realization of controlled fusion energy.

The most interesting and challenging question raised by the above experiments is whether the current reversal equilibrium configurations (CRECs) exist or not, which has resulted in active theoretical research activities [4–7].

In Ref. [7], it was numerically demonstrated that CRECs really exist. However, in Ref. [7], only force-free equilibria with zero plasma pressure gradient were considered, and it was also assumed that the toroidal current density could be discontinuous. Therefore, it is of interest to investigate whether CRECs exist for a more realistic situation without the force-free assumption and the discontinuous current density assumption.

In this Letter, we propose an analytical theoretical model for CRECs, and we shall demonstrate that CRECs exist with finite radial gradient of plasma pressure and continuous toroidal current density and that current density reversal is accompanied by pressure gradient reversal.

We begin with the Grad-Shafranov equation of tokamak equilibrium,

$$\left(x\partial_x \frac{1}{x}\partial_x + \partial_z^2\right)\psi = -\frac{1}{2}x^2\frac{d\beta}{d\psi} - \frac{1}{2}\frac{dg^2}{d\psi} = -xj_\phi, \quad (1)$$

where $\psi = \Psi/B_0a^2$ is the normalized poloidal magnetic flux; $x = R/a$, $z = Z/a$, $\beta(\psi) = 2\mu_0 p(\psi)/B_0^2$, $g(\psi) = F(\psi)/B_0a$, with magnetic field represented by $\vec{B} = F(\psi)\nabla\phi + \nabla\Psi \times \nabla\phi$; ϕ is the ignorable angle in the cylindrical coordinate system (R, ϕ, Z). $p(\psi)$ is the plasma pressure; B_0 is the vacuum magnetic field evaluated at $R = R_0$ $(x = x_0)$; R_0 and a are the major radius and the minor radius, respectively. $j_\phi = J_\phi\mu_0a/B_0$, and J_ϕ is the toroidal current density.

To investigate the CRECs with finite pressure and continuous current density, we propose the following analytical model,

$$-\frac{1}{2}\frac{d\beta}{d\psi} = a_1, \quad (2a)$$

$$-\frac{1}{2}\frac{dg^2}{d\psi} = -a_2 - \alpha^2\psi. \quad (2b)$$

Consider the circular cross section. And make the transform from the cylindrical coordinate system (x, ϕ, z) to the toroidal coordinate system (r, θ, ϕ), with $x = x_0 + r\cos\theta$, $z = r\sin\theta$. Without loss of generality, we take the boundary condition as $\psi(r = 1) = 0$.

We have found that it is convenient to consider the toroidal vector potential instead of the poloidal flux. Setting

$$\psi = xA, \quad (3)$$

we found that

$$\left(\frac{1}{x}\partial_x x\partial_x + \partial_z^2 - \frac{1}{x^2}\right)A + \alpha^2A = a_1x - a_2\frac{1}{x}, \quad (4a)$$

$$A(r = 1) = 0. \quad (4b)$$

Equation (4) is identified as a nonhomogeneous Helmholtz equation in cylindrical geometry. This equation can be solved by the standard method of expansion with respect to eigenfunctions [8].

Consider a large-aspect-ratio tokamak ($x_0 \gg 1$). Taking the ordering $\psi \sim \mathcal{O}(1)$, $\beta \sim \mathcal{O}(1/x_0^2)$, we have $A \sim \mathcal{O}(1/x_0)$, $a_1 \sim \mathcal{O}(1/x_0^2)$. Transforming Eq. (4) to the toroidal coordinate system (r, θ, ϕ), to $\mathcal{O}(1/x_0^2)$ we have

$$\left(\frac{1}{r}\partial_r r\partial_r + \frac{1}{r^2}\partial_\theta^2\right)A + \alpha^2 A + \frac{\cos\theta}{x_0}\partial_r A - \frac{\sin\theta}{x_0}\frac{1}{r}\partial_\theta A = a_1 x - a_2\frac{1}{x}. \quad (5)$$

To solve the corresponding homogeneous equation, we make the Fourier expansion

$$A(r,\theta) = \sum_{m=0}^{M} A_m(r)\cos m\theta. \quad (6)$$

The eigenvalue problem is reduced to

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}A_m - \frac{m^2}{r^2}A_m + \lambda^2 A_m = -\frac{1}{2x_0}\left(\frac{d}{dr}+\frac{m+1}{r}\right)A_{m+1} - \frac{1+\delta_m^1}{2x_0}\left(\frac{d}{dr}-\frac{m-1}{r}\right)A_{m-1}, \quad (7)$$

where $A_{0-1} = 0 = A_{M+1}$ is understood. Solution to Eq. (7) is readily found by making the expansion

$$A_m = \sum_{k=1}^{K} C_{m,k} J_m(\mu_{m,k} r), \quad (8)$$

where J_m are Bessel functions, and $J_m(\mu_{m,k}) = 0$.

Substituting Eq. (8) into Eq. (7), through a lengthy but straightforward mathematical manipulation, we found

$$C_{m,k} = \sum_{i=1}^{K} C_{m-1,i} D_{k,i}^{(m-)} + \sum_{i=1}^{K} C_{m+1,i} D_{k,i}^{(m+)}, \quad (9)$$

$$D_{k,i}^{(m-)} = \frac{1}{\lambda^2 - \mu_{m,k}^2}\frac{\mu_{m,k}J_{m-1}(\mu_{m,k})}{J_{m,k}} \times \frac{\mu_{m-1,i}J_m(\mu_{m-1,i})}{\mu_{m-1,i}^2 - \mu_{m,k}^2}\frac{1+\delta_m^1}{2x_0}, \quad (10a)$$

$$D_{k,i}^{(m+)} = \frac{1}{\lambda^2 - \mu_{m,k}^2}\frac{\mu_{m,k}J_{m-1}(\mu_{m,k})}{J_{m,k}} \times \frac{\mu_{m+1,i}J_m(\mu_{m+1,i})}{\mu_{m+1,i}^2 - \mu_{m,k}^2}\frac{1}{-2x_0}, \quad (10b)$$

where $D_{k,i}^{(0-)} = 0 = D_{k,i}^{(M+)}$ is understood and $J_{m,k} = (1/2)[J_{m+1}(\mu_{m,k})]^2$.

Equation (9) is the algebraic form of the eigenvalue problem. From Eq. (10), it is not hard to understand that the eigenvalues are close to $\mu_{m,k}$. The eigenvalue close to $\mu_{m,k}$ shall be denoted as $\lambda_{m,k}$ and the corresponding eigenvector shall be denoted as $C_{n,i}^{m,k}$ ($n = 0, 1, \ldots, M$; $i = 1, 2, \ldots, K$). This eigenvalue problem is numerically solved; it is found that the dominating component of the eigen-vector $C_{n,i}^{m,k}$ ($n = 0, 1, \ldots, M$; $i = 1, 2, \ldots, K$) is $C_{n=m,i=k}^{m,k}$.

The eigenfunction corresponding to the eigenvalue $\lambda_{m,k}$ is

$$A_{m,k}^E(r,\theta) = \sum_{n=0}^{M} E_n^{m,k}(r)\cos n\theta, \quad (11)$$

$$E_n^{m,k}(r) = \sum_{i=1}^{K} C_{n,i}^{m,k} J_n(\mu_{n,i} r). \quad (12)$$

With the eigenvalue problem solved, Eq. (5) is readily solved by the standard method [8]. Expand with respect to the eigenfunctions,

$$A = \sum_{m=0}^{M}\sum_{k=1}^{K} d_{m,k} A_{m,k}^E. \quad (13)$$

The coefficients are given by

$$d_{m,k} = \frac{1}{\alpha^2 - \lambda_{m,k}^2}\frac{L_{m,k}}{N_{m,k}}, \quad (14a)$$

$$L_{m,k} = \int_0^{2\pi} d\theta \int_0^1 xrdrA_{m,k}^E\left(a_1 x - a_2\frac{1}{x}\right), \quad (14b)$$

$$N_{m,k} = \int_0^{2\pi} d\theta \int_0^1 xrdr(A_{m,k}^E)^2. \quad (14c)$$

It is straightforward to show that

$$L_{m,k} = \sum_{k=1}^{K} L_i^{m,k}, \quad (15a)$$

$$L_i^{m,k} = 2\pi\left\{(a_1x_0^2 - a_2)\frac{J_1(\mu_{0,i})}{\mu_{0,i}} + a_1\frac{1}{\mu_{0,i}}\left[\frac{1}{\mu_{0,i}}J_2(\mu_{0,i}) - \frac{1}{2}J_3(\mu_{0,i})\right]\right\}C_{0,i}^{m,k} + 2\pi a_1 x_0\frac{J_2(\mu_{1,i})}{\mu_{1,i}}C_{1,i}^{m,k} + \frac{\pi}{2}a_1\frac{J_3(\mu_{2,i})}{\mu_{2,i}}C_{2,i}^{m,k}. \quad (15b)$$

$$N_{m,k} = \int_0^1 \pi rdr\{x_0[(E_0^{m,k})^2 + \sum_{n=0}^{M}(E_n^{m,k})^2] + r[E_0^{m,k}E_1^{m,k} + \sum_{n=0}^{M-1}E_n^{m,k}E_{n+1}^{m,k}]\} \quad (16a)$$

$$\approx \frac{\pi}{2}x_0(1+\delta_m^0)[C_{m,k}^{m,k}J_{m+1}(\mu_{m,k})]^2, \quad (16b)$$

where in Eq. (16b) some small terms have been dropped.

It should be pointed out that the solution to the Grad-Shafranov equation together with the model profile, Eq. (2), presented above is correct to $\mathcal{O}(1/x_0^2)$, with x_0 the aspect ratio. In our analytical solution, it is clear that the toroidicity-induced poloidal harmonic coupling is included not only through the eigenvalue problem described in Eq. (7) but also through the geometrical factor x in Eq. (3) and Eq. (14b).

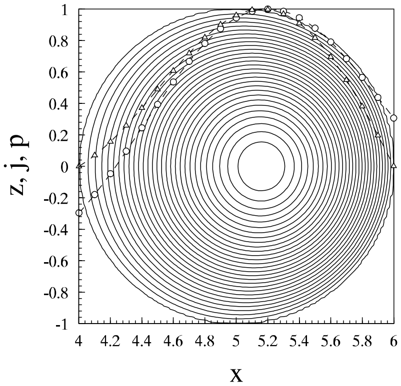

FIG. 1. Magnetic flux surfaces, normalized current density (open circles) and normalized pressure (open triangles) for a normal configuration with $I_N(1) = 0.33$, $I_N(0.7) = 0.50$, and $\beta_V = 0.01$. $\alpha = \lambda_{0,1} - 0.06960$, $(\lambda_{0,1} = 2.406904)$, $a_1 = -0.04531$, and $a_2 = -1.0808$..

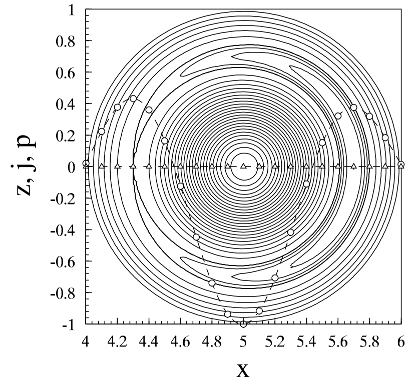

FIG. 2. Magnetic flux surfaces, normalized current density (open circles) and normalized pressure (open triangles) for a force-free CREC with $I_N(1) = 0.25$, $I_N(0.7) = 0.00$, and $\beta_V = 0.02$. $\alpha = \lambda_{0,2} - 0.04402$, $(\lambda_{0,2} = 5.520984)$, $a_1 = 0.0$, and $a_2 = 0.05852$.

Now we are at the position to make a brief comparison between our solution to a previous work [9], where Eq. (2a) and (2b) was used. First, the main physics of the present work is the CREC that was not discussed in Ref. [9]. Second, although the differential equation [Eq. (10) there] in Ref. [9] is equivalent to our Eq. (4a) and hence it appears to be that Ref. [9] solved the same nonhomogeneous Helmholtz equation, the unusual boundary condition chosen there made the problem essentially a homogeneous problem. Finally, Ref. [9] gave an exact solution to the eigenvalue problem, as we have indicated, by using the cylindrical coordinate system; the reason we chose the toroidal coordinate system is that it clearly reveals the toroidicity-induced poloidal harmonic coupling, which is important in CRECs, as has been discussed in the last paragraph; using cylindrical coordinate system, we have obtained an exact solution to Eq. (4a) with rectangular cross section and zero ψ boundary condition; the results show similar CRECs, which shall be published elsewhere.

Note that there are three free parameters, a_1, a_2, and α in the solution. These three parameters can be determined by the total current, the current inside a finite central region ($r \leq r_0$), and the plasma pressure, as will be discussed in the following.

Define the normalized current

$$I_N(r) \equiv \frac{\mu_0 I x_0}{2\pi a r^2 B_0} = -\frac{1}{2\pi r^2}\int_0^r r dr d\theta (a_1 x^2 - a_2 - \alpha^2 \psi), \quad (17)$$

where I is the plasma current. Note that I_N is roughly inverse of the safety factor. And the volume-averaged beta-value is

$$\beta_V = \beta_0 - \frac{2}{\pi} a_1 \int_0^1 \frac{x}{x_0} r dr d\theta \psi, \quad (18)$$

where β_0 is chosen so that the minimum of the plasma pressure ($\beta = \beta_0 - 2a_1\psi$) is zero.

It is not hard to show that

$$I_N(r) = \alpha^2 \sum_{m=0}^{M}\sum_{k=1}^{K} d_{m,k} I^{\alpha}_{m,k} - \frac{1}{2}(a_1 x_0^2 - a_2) - \frac{1}{8} a_1 r^2, \quad (19)$$

$$I^{\alpha}_{m,k} = \frac{x_0}{r}\sum_{i=1}^{K}\frac{J_1(\mu_{0,i} r)}{\mu_{0,i}} C^{m,k}_{0,i} + \frac{1}{2}\sum_{i=1}^{K}\frac{J_2(\mu_{1,i} r)}{\mu_{1,i}} C^{m,k}_{1,i}; \quad (20)$$

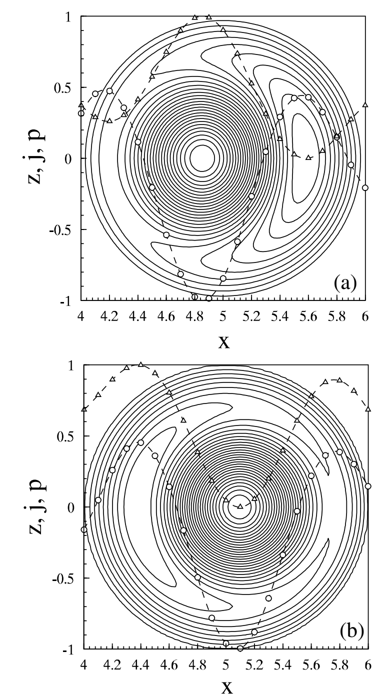

FIG. 3. Magnetic flux surfaces, normalized current density (open circles) and normalized pressure (open triangles) for a CREC with $I_N(1) = 0.25$, $I_N(0.7) = 0.00$, and $\beta_V = 0.02$. (a) $\alpha = \lambda_{0,2} - 0.08818$, $a_1 = 0.1193$, $a_2 = 3.1034$. (b) $\alpha = \lambda_{0,2} - 0.02088$, $a_1 = -0.06325$, and $a_2 = -1.5527$.

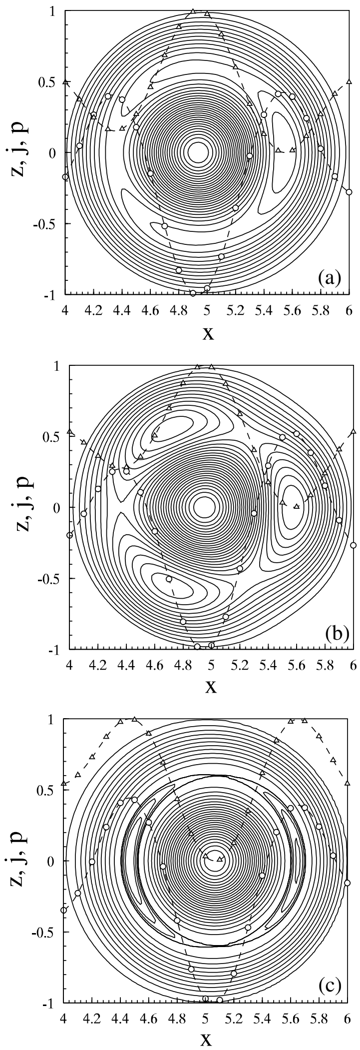

FIG. 4. Magnetic flux surfaces, normalized current density (open circles) and normalized pressure (open triangles) for a CREC with $I_N(1) = 0.25$, $I_N(0.6) = 0.00$, and $\beta_V = 0.00$. (a) $\alpha = \lambda_{31} - 0.00586, (\lambda_{31} = 6.379377)$, $a_1 = 0.09234$, $a_2 = 0.1721$. (b) $\alpha = \lambda_{3,1} - 0.00102$, $a_1 = 0.0758$, and $a_2 = -0.2507$. (c) $\alpha = \lambda_{3,1} + 0.03504$, $a_1 = -0.04167$, and $a_2 = -3.2465$.

$$\beta_V = \beta_0 - 4a_1 \sum_{m=0}^{M} \sum_{k=1}^{K} d_{m,k} \sum_{i=1}^{K} B_{m,k}^i, \tag{21}$$

$$B_{m,k}^i = \left\{ x_0 \frac{J_1(\mu_{0,i})}{\mu_{0,i}} + \frac{1}{x_0 \mu_{0,i}} \left[\frac{1}{\mu_{0,i}} J_2(\mu_{0,i}) - \frac{1}{2} J_3(\mu_{0,i}) \right] \right\} \times C_{0,i}^{m,k} + \frac{J_2(\mu_{1,i})}{\mu_{1,i}} C_{1,i}^{m,k} + \frac{1}{4x_0} \frac{J_3(\mu_{2,i})}{\mu_{2,i}} C_{2,i}^{m,k}. \tag{22}$$

An equilibrium configuration is determined by specifying the total current $I_N(r = 1) = I_{N1}$, the current inside a finite central region $I_N(r = r_0) = I_{N0}$, and the volume-averaged beta-value $\beta_V = \beta_{V0}$. With a solution $\psi = \psi_0$ given by $\alpha = \alpha_0$, $a_1 = a_{10}$, and $a_2 = a_{20}$ satisfying $I_N(1) = I_{N10}$, $I_N(r_0) = I_{N00}$, and $\beta_V = \beta_{V0}$, a new solution $\psi = k\psi_0$ with the same shape of flux surfaces satisfying $I_N(1) = kI_{N10}$, $I_N(r_0) = kI_{N00}$, and $\beta_V = k^2\beta_{V0}$ can be found by setting $\alpha = \alpha_0$, $a_1 = ka_{10}$, and $a_2 = ka_{20}$.

We found that a variety of equilibrium configurations could be generated by the above formalism. A few typical numerical examples will be presented in the following.

In Fig. 1, a normal configuration is shown. In Fig. 2, a force-free CREC is shown. In Fig. 3 and Fig. 4, CRECs with finite radial gradient of pressure are shown. From Fig. 3 and Fig. 4, it is seen that more than one solutions may be found for a given set of $I_N(1)$, $I_N(r_0)$, and β_V. It is also seen from Fig. 3 and Fig. 4 that the current density reversal is accompanied by pressure gradient reversal, as is consistent with Ref. [6]. For the numerical examples presented above, $x_0 = 5$, $|\psi| \leq 0.3$. It should be pointed out that in the region with positive radial gradient of pressure the plasma transport should be convection-dominated.

In conclusion, we have found that with a model current profile proposed, the Grad-Shafranov equation can be reduced to the Helmholtz equation, which can describe a variety of equilibrium configurations including the CRECs with finite radial gradient of plasma pressure and continuous current density. With the eigenvalue problem solved in toroidal coordinate system, we have found an analytical solution to the Grad-Shafranov equation; this solution has demonstrated that CRECs exist with finite radial gradient of plasma pressure and continuous current density, and that current density reversal is accompanied by pressure gradient reversal. Finally, it should be pointed out that recent numerical simulations found a reverse current could not be sustained because of a tearing type instability [10]; the relevance of the CRECs model proposed in this Letter to experiments is still an open question.

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