Statistics of Defect-Mediated Turbulence Influenced by Noise

Hongli Wang
Department of Physics, Peking University, Beijing 100871, People's Republic of China
(Received 22 April 2004; published 5 October 2004)

The influence of white noise on defect-mediated turbulence which is modeled by the complex Ginzburg-Landau equation is investigated. We show that the dynamics of defects in the noise-driven spatiotemporal chaos can be described by a simple statistical model. The noise enhances significantly the ability of the turbulent background to advocate new defects with a constant rate, and at the same time it increases the vanishing of defects in the system by introducing an additional annihilation rate that is proportional to the number of defects. A universal probability distribution function is derived for the number of defect pairs.

DOI: 10.1103/PhysRevLett.93.154101 PACS numbers: 05.45.Jn, 02.50.Ga, 05.40.Ca

Noises often play nontrivial roles in nonlinear dynamics, and their influence on low-dimensional systems has been investigated extensively in the last 20 years [1–5]. Important examples include noise-induced transitions and particularly the phenomenon of stochastic resonance. In a more recent past, investigations of the effect of noise have been extended to high-dimensional spatially distributed systems [4]. Much more fruitful and complicated phenomena such as noise-induced fronts [6], phase transitions [7], noise-sustained or induced spatiotemporal patterns [8], and spatiotemporal stochastic resonances [9] were found. While most studies along this line have been devoted exclusively to dynamical systems that exhibit ordered patterns, the interplay between noise and chaotic dynamics might be most interesting because they represent, respectively, two distinct kinds of essentially different irregularities. The noise is generated by genuine stochastic processes; the randomness of chaos is, however, of a pseudo kind and is deterministic in essence. The effect of noise on low-dimensional chaos has been extensively studied [10]; the interaction between noise and high-dimensional spatiotemporal chaos has, however, been seldom touched upon [11].

In this Letter, we report the effects of noise on an important class of spatiotemporal chaos, i.e., defect-mediated turbulence which is at the focus of experimental and theoretical studies [12]. Defect turbulence has been found to be abundant in systems such as autocatalytic chemical reactions [13], fluid convection [14], cardiac tissue [15], electroconvection in liquid crystals [16], nonlinear optics [17], Langmuir circulation in the oceans [18], and so on. A striking feature of the dynamics in these very different systems is that they can be characterized by a hopefully universal description based only on the defect dynamics. A first probabilistic model for defect-mediated turbulence that describes fluctuations of the number of defects was given by Gil *et al* [19]. They proposed a Poisson-like distribution for the number of defects which was found to be quite general for oscillatory systems. The theoretical prediction has been proved in the simulations of the complex Ginzburg-Landau equation (CGLE) [19] and experimentally in electroconvection of nematic liquid crystals [16]. Recent experiments of defect turbulence in inclined layer convection also agree with the theoretical prediction when boundary effects are taken into account [20]. The distribution matches well with the most recent simulation study of defect-mediated turbulence in media where the underlying local dynamics is chaotic [21].

When external noises are introduced into defect-mediated turbulence, the picture is anticipated to be definitely changed due to the chaotic nature of turbulence. Under appropriate intensity of noise, we find that the noise enhances the fluctuation background to advocate new defects with an additional rate which is independent of the number of defect pairs, while at the same time it induces an additional annihilation rate that is proportional to defect pairs. We derive here a universal probability distribution function for defects.

We model in the following the defect-mediated turbulence with CGLE [22], which is the prototype for oscillatory media, and introduce additively into CGLE a noise term in the following form:

$$\partial_t A = A + (1 + i\alpha)\nabla^2 A - (1 + i\beta)|A|^2 A + (1 + i)\xi(r, t), \tag{1}$$

where $\xi(r, t)$ is a Gaussian distributed stochastic real field of white noise that has the property

$$\langle \xi(r, t)\xi(r', t')\rangle = 2D\delta(t - t')\delta(r - r'), \tag{2}$$

where D is the intensity of noise. Equation (1) does not have a direct physical correspondence but represents reasonably a theoretical model for investigating the interaction between noise and spatiotemporal chaos. We simulate Eq. (1) in a two-dimensional domain of size $L \times L$ numerically using periodic boundary conditions. Spatiotemporal chaos in CGLE has been studied extensively [22,23], and we in the following fix $\alpha = 1.2$, $\beta = -1.3$ and focus our attention on the specific chaotic state of defect-mediated turbulence in order to investigate the effect of noise.

A defect in the turbulent pattern corresponds to a local position in the medium where the amplitude $|A| = 0$ and the phase is undefined. It is characterized by its topological charge defined by $\frac{1}{2\pi}\oint \nabla\phi(r,t)dl = m_{\rm top}$, where $\phi(r,t)$ is the local phase and the integral is calculated along a closed curve surrounding the defect. The charge $m_{\rm top}$ takes typically $+1$ or -1. A defect is identified where the contours $\mathrm{Re}(A) = 0$ and $\mathrm{Im}(A) = 0$ intersect. The number of defects in the turbulent pattern can be counted as how many times the contours intersect in the whole domain. When CGLE is free of noise ($D = 0$), the behavior of defects can be considered as a Markov process [19]. The creation rate Ξ_+ is independent of the current number of defects in the system, i.e., $\Xi_+(n) = C_0$, while the annihilation Ξ_- is proportional to the square of defect pairs, $\Xi_-(n) = A_0 n^2$. The probability distribution function for the number of defect pairs n is the squared-Poisson distribution, $P(n) = \gamma^n/[I_0(\sqrt{\gamma})(n!)^2]$, where $\gamma = C_0/A_0$, and I_0 is the Bessel function.

We turn on the noise and adjust noise intensity D. A not too strong noise does not change the fundamental picture of defect-mediated turbulence. In the noise-driven system, the defects survive and turn up and wither away spontaneously in the pattern. Figure 1(b) depicts a snapshot of the turbulent field with $D = 0.01$, which is similar to the picture of Fig. 1(a) with $D = 0$. As D is increased to a sufficiently strong intensity, the vortex in the turbulence is ruined and the pattern is smeared as can be seen in Fig. 1(c). Trajectories of topological defects show how the noise affects the diffusion of defects. Figs. 1(d)–1(f) demonstrate the defect paths over a short period of time under different intensities of noise. In the case of weak noise [Fig. 1(e)], the number of trajectories is increased by an amount, but the paths are still regular and clear. Blurred and irregular trajectories are induced when a strong noise is applied [Fig. 1(f)].

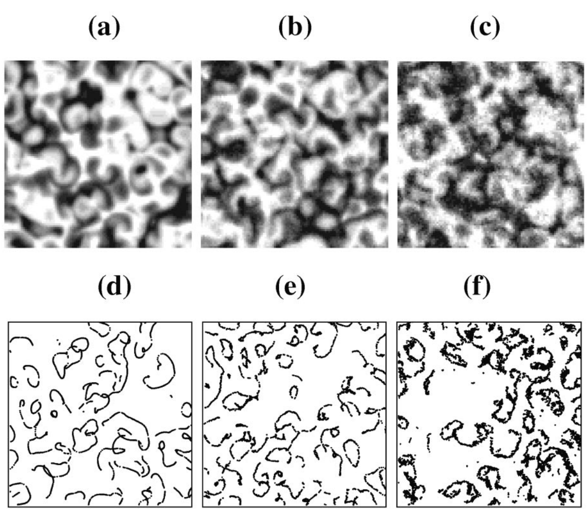

FIG. 1. Impact of white noise on defect-mediated turbulence with different noise intensities. Gray-scaled plots of (a)–(c) depict $\mathrm{Re}(A)$; (d)–(f) show trajectories of topological defects over a period of 20 temporal units. Parameters are $D = 0.0$, 0.01, and 0.05 for (a),(d), (b),(e), and (c),(f), respectively. Parameters are $\alpha = 1.2$, $\beta = -1.3$, and domain size $L = 100$.

For $D = 0$, topological defects are created and annihilated strictly in pairs of opposite charge because the net topological charge is conserved and equal to zero due to periodic boundary conditions. This can be made more clear by the contours that are used to locate the defects. As shown in Fig. 2, the contours $\mathrm{Re}(A) = 0$ and $\mathrm{Im}(A) = 0$ are all closed curves (due to the periodic boundary condition). As the pattern evolves, the contour lines are constantly created and blow up. They continuously change their shape, merge with each other, or shrink and disappear. Every time the two types of contours intersect or detach at a local position, a pair of two nearby intersection points, which correspond to a pair of topological defects of opposite charge, is created or annihilated. The number of the intersection points of contours is always even, half of which are defects with $m_{\rm top} = +1$ while the other half have charge $m_{\rm top} = -1$. Defects of opposite charges distribute alternately every other one along the closed contours. In the presence of noise, the contours are still closed curves (right panel, Fig. 2). Therefore, even in the presence of noise, defects are created or annihilated strictly by pairs. The noise has, however, influenced the rates of defect creation and annihilation.

The creation and annihilation rates as functions of defect pairs are shown in Fig. 3. They are calculated by counting the current defects and those created and that vanish in a subsequent time unit and are then averaged during the evolution. The creation of defects is still independent of the number of defect pairs as in the noise free case but has been increased significantly to a much higher level [Fig. 3(a)]. The noise has therefore enhanced the fluctuation background to advocate new defects with an additional rate. The annihilation rate is also drastically affected. Defects vanish with a much higher rate under the impact of noise [Fig. 3(b)].

On the basis of the simulation results in Fig. 3, we can extend the model given by Gil *et al* [19] in order to describe the dynamics of defects in the presence of noise.

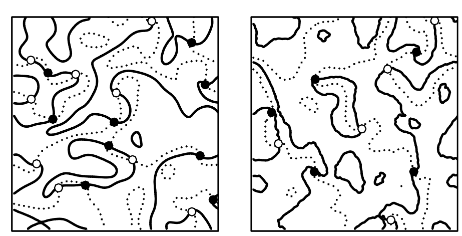

FIG. 2. Contours of $\mathrm{Re}(A) = 0$ (solid line) and $\mathrm{Im}(A) = 0$ (dotted line) without (left panel) and with (right panel, $D = 0.01$) the influence of noise. The lines are closed curves when periodic boundary is taken into account. Black dots are for defects with charge $+1$, open circles for defects with charge -1. Domain size $L = 50$.

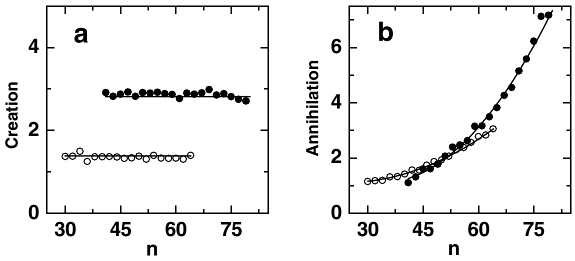

FIG. 3. Creation rates (a) and annihilation rates (b) of defects as a function of defect pairs n with (black circle) and without (open circle) the influence of noise. The solid lines in (a) and (b) correspond to constant creation rates and annihilation rates that are approximately proportional to quadratic polynomials, respectively. Domain size $L = 100$.

We anticipate that the noise has increased the amount of phase instability in the system which is responsible for the creation of defects. The phase gradient under the action of stochastic fluctuations of noise has led to a faster pinching of the equiphases. The relaxation of the field leads thereby to a faster creation of defects [Fig. 3(a)]. Notice that even though the defects are created in pairs, the creation rate is not a proportion to the square of defects. The noise can have influenced the detailed dynamic process of the contours in their creation, shape distortion, merging, and vanishing. The annihilation of defects can therefore be also influenced. Simulation results [Fig. 3(b)] indicate that the annihilation rate scales approximately with quadratic polynomials. It is reasonable to assume that the new creation rate $\Xi_+(n)$ and the new annihilation rate $\Xi_-(n)$ in the presence of noise generally take the following forms:

$$\Xi_+(n) = C_0 + C', \tag{3}$$

$$\Xi_-(n) = A_0 n^2 + A'n, \tag{4}$$

where the constant C' and the linear term $A'n$ are induced by the noise. One sees that although defects are created and annihilated both by pairs, the rates do not definitely scale with n^2. This suggests that topological defects in defect-mediated turbulence probably cannot be considered as a "defect gas" and cannot be described by a chemical-like kinetics. The linear term assumed in Eq. (4) is introduced on the base of numerical findings [Fig. 3(b)], and the validity of Eqs. (3) and (4) is further justified by a close agreement between the subsequent theoretical predictions and simulation data.

In the stationary state of detailed balance, the probability distribution $P(n)$ for the number of defects satisfies the master equation,

$$P(n) = \frac{\theta}{\zeta n + n^2} P(n - 1), \tag{5}$$

where $\theta = (C_0 + C')/A_0$, $\zeta = A'/A_0$. A simple recursion manipulation leads to the following modified Poisson distribution which has also been derived in [20] in a different circumstance

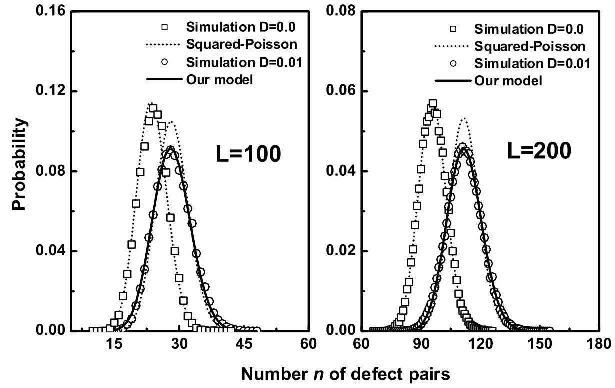

FIG. 4. Normalized distributions of simulation with $D = 0.0$ (square) and $D = 0.01$ (circle) with $L = 100$ and $L = 200$, respectively. The dotted lines correspond to the squared-Poisson distributions, while the full lines are the theoretical prediction of Eq. (6) which agrees perfectly with the simulation data. For $L = 200$ and $D = 0.01$, we simulate $\langle n \rangle = 112$, $\langle n^2 \rangle = 12\,546.0$, and a best fitting of Eq. (6) is achieved with $\zeta = 129.0$ with $\theta = 26\,942.4$.

$$P(n) = \frac{\theta^{n+\zeta/2}}{I_\zeta(2\sqrt{\theta})\Gamma(n + 1 + \zeta)n!}, \tag{6}$$

where I_ζ is the modified Bessel function.

Figure 4 shows that the distribution of Eq. (6) matches the simulation data very well, and the squared-Poisson distribution has deviated significantly from the data. When applying the modified Poisson distribution to fit the simulation data, the relation $\theta = \zeta\langle n \rangle + \langle n^2 \rangle$ holds and can be utilized. The distribution of Eq. (6) captures also correctly the mean and variance of the simulation data. Figure 5 shows good agreement between the model (underlined characters) and the simulation result (regular characters) when the intensity of the noise is tuned in the range between $D = 0.0$ and $D = 0.03$. The squared-

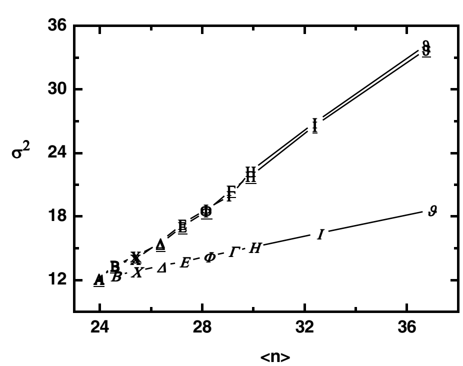

FIG. 5. The variance $\sigma^2 = \langle n^2 \rangle - \langle n \rangle^2$ against the mean $\langle n \rangle$ when intensity of noise D is adjusted between 0.0 and 0.03. Results of Eq. (6) (underlined characters) are in close accordance with the simulation data (regular characters). The italic characters correspond to results of the squared-Poisson distribution. The lines are to guide the eye.

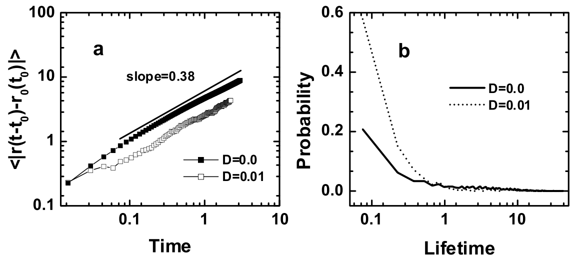

FIG. 6. (a) Influence of noise on diffusive property of defects. $\langle|r(t-t_0)-r_0(t_0)|\rangle$ is calculated with trajectories of a bundle of long-lived defects. The solid line has slope 0.38. (b) Normalized distribution of lifetime of defects without (solid line) and with (dotted line, $D = 0.01$) the influence of noise.

Poisson distribution (italic characters) fails obviously in the situation when defect-mediated turbulence is driven by noise. As the noise intensity is increased, both the creation and the annihilation rates grow monotonously.

We also checked the impact of noise on the diffusion of defects. The quantity $\langle|r(t-t_0)-r_0(t_0)|\rangle$, where r_0 is the local position where a defect is created at t_0, as function of time is calculated and shown in Fig. 6(a). For Brownian motion, this quantity grows proportional to t^δ with $\delta = 1/2$. The exponent δ calculated in Fig. 6(a) is about 0.38, which is less than $1/2$ and is therefore subdiffusive. Figure 6(b) depicts the probability distribution of defect lifetime without and with the influence of noise. One sees that the noise has decreased drastically the lifetimes of defects.

To summarize, we have shown that the dynamics of defects in defect-mediated turbulence driven by noise can be described by a simple statistical model. The noise has enhanced the ability of the turbulent background to create defects with a constant rate while at the same time it destroys the existing defects at a rate that is proportional to defect pairs in the system. For defect-mediated spatiotemporal chaos, researchers have proposed that the macroscopic behavior of defect turbulence can be understood by a simpler description of the dynamics of defects instead of the original detailed partial different equations [24]. A measure of the complexity of defect-mediated turbulence was shown to be proportional to the average number of defects in the system [25]. We demonstrated here that even when defect chaos is driven by noise, the complicated system can still be understood statistically by taking into account the effect of noise on the creation and annihilation of defects. Our results for the effects of noise on defect turbulence and the probability distribution function [Eq. (6)] for the number of defects should be generic in oscillatory systems because the CGLE is generic and the defect-mediated spatiotemporal chaos it describes represents a universal class of turbulence.

The work is supported by the Natural Science Foundation of China (Grant No. 10204002) and the Special Funds for Major State Basic Research Project of China.

[1] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984).
[2] K. Lindenberg, B. J. West, and J. Masoliver, *Noise in Nonlinear Dynamical Systems* (Cambridge University Press, Cambridge, 1989).
[3] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[4] J. Garcia-Ojalvo and J. M. Sancho, *Noise in Spatial Extended Systems* (Springer, New York, 1999).
[5] B. Lindner *et al.*, Phys. Rep. **392**, 321 (2004).
[6] J. Garcia-Ojalvo and J. M. Sancho, Phys. Rev. E **49**, 2769 (1994); M. A. Santos and J. M. Sancho, Phys. Rev. E **59**, 98 (1999); D. Panja and W. van Saarloos, Phys. Rev. E **66**, 036206 (2002).
[7] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994); A. Becker and L. Kramer, Phys. Rev. Lett. **73**, 955 (1994); S. Kim, S. H. Park, and C. S. Ryu, Phys. Rev. Lett. **78**, 1616 (1997).
[8] See, for example, S. Kadar, J. Wang, and K. Showalter, Nature (London) **391**, 770 (1998); I. Sendina-Nadal *et al.*, Phys. Rev. Lett. **84**, 2734 (2000); S. Rudiger *et al.*, Phys. Rev. Lett. **90**, 128301 (2003).
[9] P. Jung and G. Mayer-Kress, Phys. Rev. Lett. **74**, 2130 (1995); J. M. G. Vilar and J. M. Bubi, Phys. Rev. Lett. **78**, 2886 (1997); L. Q. Zhou, X. Jia, and Q. Ouyang, Phys. Rev. Lett. **88**, 138301 (2002); X. Jia, L. Q. Zhou, and Q. Ouyang, Chin. Phys. Lett. **21**, 435 (2004).
[10] J. P. Crutchfield, J. D. Farmer, and B. A. Huberman, Phys. Rep. **92**, 45 (1982).
[11] H. Wang and Q. Ouyang, Phys. Rev. E **65**, 046206 (2002).
[12] For the most recent reports, refer to Y. N. Young and H. Riecke, Phys. Rev. Lett. **90**, 134502 (2003); W. Y. Woon and Lin I, Phys. Rev. Lett. **92**, 065003 (2004).
[13] Q. Ouyang and J. M. Flesselles, Nature (London) **379**, 143 (1996).
[14] S.W. Morris, E. Bodenschatz, D. S. Cannell, and G. Ahlers, Phys. Rev. Lett. **71**, 2026 (1993).
[15] J. M. Davidenko, A.V. Pertsov, R. Salomonsz, W. Baxter, and J. Jalife, Nature (London) **355**, 349 (1993).
[16] I. Rehberg, S. Rasenat, and V. Steinberg, Phys. Rev. Lett. **62**, 756 (1989).
[17] P. Ramazza, S. Residori, G. Giacomelli, and F. Arecchi, Europhys. Lett. **19**, 475 (1992).
[18] T. M. Haeusser and S. Leibovich, Phys. Rev. Lett. **79**, 329 (1997).
[19] L. Gil, J. Lega, and J. L. Meunier, Phys. Rev. A **41**, 1138 (1990).
[20] K. E. Daniels and E. Bodenschatz, Phys. Rev. Lett. **88**, 034501 (2002); K. E. Daniels and E. Bodenschatz, Chaos **13**, 55 (2003).
[21] J. Davidsen and R. Kapral, Phys. Rev. Lett. **91**, 058303 (2003).
[22] I.S. Aranson and L. Kramer, Rev. Mod. Phys. **74**, 99 (2002).
[23] H. Chate and P. Manneville, Physica (Amsterdam) **224A**, 348 (1996).
[24] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993); R. E. Ecke, Y. Hu, R. Mainieri and G. Ahlers, Science **269**, 1704 (1995).
[25] D. A. Egolf, Phys. Rev. Lett. **81**, 4120 (1998).