Prethermalization

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Prethermalization of the equation of state and the kinetic temperature to their equilibrium values occurs on time scales dramatically shorter than the thermal equilibration time. This is a crucial ingredient for the understanding of collisions of heavy nuclei or other nonequilibrium phenomena in complex quantum and classical many body systems. We also compare the chemical equilibration time with other characteristic time scales.

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Prethermalization is a universal far-from-equilibrium phenomenon which describes the very rapid establishment of an almost constant ratio of pressure over energy density (equation of state), as well as a kinetic temperature based on average kinetic energy. The phenomenon occurs on time scales dramatically shorter than the thermal equilibration time. As a consequence, prethermalized quantities approximately take on their final thermal values already at a time when the occupation numbers of individual momentum modes still show strong deviations from the late-time Bose-Einstein or Fermi-Dirac distribution.

The abundance of experimental data on matter in extreme conditions from relativistic heavy-ion collision experiments, as well as applications in astrophysics and cosmology urge a quantitative understanding of nonequilibrium dynamics in quantum field theories. Collision experiments seem to indicate early thermalization, whereas the present theoretical understanding of QCD suggests a much longer thermal equilibration time. For example, the successful application of hydrodynamics already less than 1 fm after the collision is so far unexplained from theory [1].

To resolve these questions, it is important to understand to what "degree" thermalization is required to explain the observations. Different quantities effectively thermalize on different time scales and a complete thermalization of all quantities may not be necessary. For instance, an approximately time-independent equation of state $p = p(\epsilon)$, characterized by an almost fixed relation between pressure p and energy density ϵ, may form very early—even though the system is still far from equilibrium. Such an almost constant equation of state is a crucial ingredient for the use of efficient hydrodynamic descriptions, since it is needed to close the system of equations obtained from the conservation of the energy momentum tensor.

The initial stages of a collision require one to consider quantum fields in extreme nonequilibrium situations. Connecting this far-from-equilibrium dynamics at early times with the approach to thermal equilibrium at late times is a challenge for theory. Achieving this goal is crucial for a comparison between the time scales of prethermalization and thermal equilibration, and we investigate this question here quantitatively in an effective quark-meson model. Approaches based on small deviations from equilibrium, or on a sufficient homogeneity in time underlying kinetic descriptions, are not applicable to describe the "link" between the initial and the late-time behavior. Classical field theory approximations are expected to be valid for not too late times but cannot determine the relevant time scale for the approach to quantum thermal equilibrium. Recently, it has been demonstrated [2–6] that far-from-equilibrium dynamics as well as subsequent thermalization of quantum fields can be described using efficient functional integral techniques. The description includes direct scattering as well as off-shell and memory effects. This is crucial to establish the different time scales for a loss of memory of initial conditions for certain "bulk quantities" as compared to "mode quantities" characterizing the evolution of individual momentum modes.

The observation that the nonequilibrium system looses a major part of the memory of the detailed initial conditions on a very short time scale is a robust feature of classical as well as quantum field theories. It has been observed [7] that approximate "equipartition" between bulk kinetic and potential energy occurs very rapidly. Below, this is the basis of our definition of a "kinetic temperature" T_{kin}. We see that the equation of state becomes almost constant at the same "prethermalization time" t_{pt}. A rapid approach to a slow evolution of the equation of state in classical field theories has also been observed with expanding space-time geometries [8]. The fast loss of memory for these quantities is based on the phenomenon of "dephasing" [9], which is independent of the interaction details. In contrast, "mode temperatures" (defined below) for individual momentum modes lose only part of the initial condition details on a somewhat longer time scale t_{damp} which depends on the interaction strength [10]. Still, t_{damp} is much smaller than the true thermal equilibration time t_{eq} which describes the universal rate of approach to the equilibrium values for all relevant correlation functions [2,3,6]. An even substantially larger separation of scales is observed in classical field theories [12–14] as compared [4] to the corresponding quantum theories.

In this Letter we consider the nonequilibrium evolution of quantum fields for a low-energy quark-meson model, which takes into account two quark flavors with a Yukawa coupling $\sim h$ to a scalar σ field and a triplet of pseudoscalar pions, $\vec{\pi}$. The theory corresponds to the well-known "linear σ model," which incorporates the chiral symmetries of massless two-flavor QCD. The classical action reads

$$S = \int d^4x \Big\{ \bar{\psi} i \partial\!\!\!/ \psi + \frac{1}{2}[\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \vec{\pi} \partial^\mu \vec{\pi}] + h\bar{\psi}[\sigma + i\gamma_5 \vec{\tau}\vec{\pi}]\psi - V(\sigma^2 + \pi^2) \Big\}. \tag{1}$$

We consider a quartic scalar self-interaction $V(\sigma^2 + \pi^2) = m_0^2(\sigma^2 + \pi^2)/2 + \lambda(\sigma^2 + \pi^2)^2/(4!N_f^2)$ with $N_f = 2$. The employed couplings are taken to be of order 1, and if not stated otherwise $h = \lambda = 1$. We emphasize that the main results of this Letter about prethermalization are independent of the detailed values of the couplings. Here we use the two-particle irreducible (2PI) effective action to two-loop order [15]. In Ref. [6] it has been shown that this approximation can be used to study the far-from-equilibrium dynamics as well as the late-time approach to quantum thermal equilibrium. The dynamics is solved numerically without further approximations (cf. Ref. [6] for calculational details). All quantities are given in units of the scalar thermal mass m [16].

Thermalization.—Nonequilibrium dynamics requires the specification of an initial state. Crucial questions of thermalization are how quickly the nonequilibrium system effectively looses the details about the initial conditions, and what are the characteristic stages of a partial loss of information. Thermal equilibrium keeps no memory about the time history except for the values of a few conserved charges. In Fig. 1 we show the effective occupation number density of fermion momentum modes, $n^{(f)}(t;p)$, as a function of time for three different momenta [17]. The plot shows two runs denoted as (A) and (B) with different initial conditions but the same energy density. Run (A) exhibits a high initial particle number density in a narrow momentum range around $\pm p$. This situation is reminiscent of two colliding wave packets with opposite and equal momentum. We emphasize, however, that we are considering a spatially homogeneous and isotropic ensemble with a vanishing net charge density. For run (B) an initial particle number density is employed which is closer to a thermal distribution.

One observes that for a given momentum the mode numbers of runs (A) and (B) approach each other at early times. The characteristic time scale for this approach is well described by the damping time $t_{\rm damp}(p)$ [18]. Irrespective of the initial distributions (A) or (B), we find (for $p/m \simeq 1$) $t_{\rm damp}^{(f)} \simeq 25\ {\rm m}^{-1}$ for fermions and $t_{\rm damp}^{(s)} \simeq 28\ {\rm m}^{-1}$ for scalars. In contrast to the initial rapid changes, one observes a rather slow or "quasistationary" subsequent evolution. The equilibration time $t_{\rm eq} \simeq 95\ {\rm m}^{-1}$ is substantially larger than $t_{\rm damp}$ and is approximately the same for fermions and scalars. Thermal equilibration is a collective phenomenon which is, in particular, rather independent of the momentum. In summary, mode quantities such as effective particle number distribution functions show a characteristic two-stage loss of initial conditions: after the damping time scale much of the details about the initial conditions is effectively lost. However, the system is still far from equilibrium and thermalization happens on a much larger time scale.

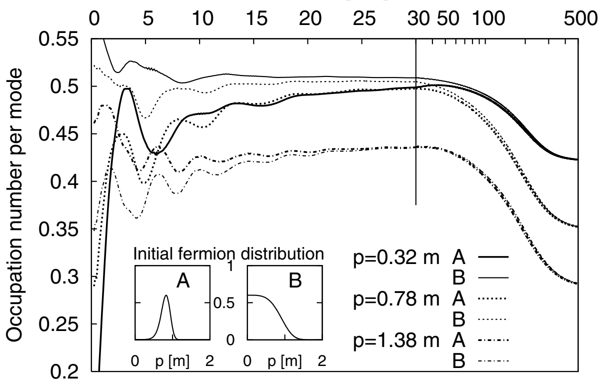

FIG. 1. Fermion occupation number $n^{(f)}(t;p)$ for three different momentum modes as a function of time. The evolution is shown for two different initial conditions with the *same* energy density. The long-time behavior is shown on a logarithmic scale for $t \geq 30\ {\rm m}^{-1}$.

We define mode temperatures $T_p^{(f,s)}(t)$ by equating the mode particle numbers $n_p^{(f,s)}(t)$ with a time and momentum dependent Bose-Einstein or Fermi-Dirac distribution, respectively [6]:

$$n_p(t) \overset{!}{=} \{\exp[\omega_p(t)/T_p(t)] \pm 1\}^{-1}. \tag{2}$$

This definition is a quantum mechanical version of its classical counterpart as defined by the squared "generalized velocities" [7]. In thermal equilibrium with $\omega_p \simeq \sqrt{p^2 + M^2}$ and $T_p = T_{\rm eq}$ Eq. (2) yields the familiar occupation numbers ($\mu = 0$). Here the mode frequency $\omega_p^{(f,s)}(t)$ is determined by the peak of the spectral function for a given time and momentum, as detailed in Ref. [6]. In Fig. 2 we show the fermion and scalar mode temperature as a function of momentum for various times $t \gg t_{\rm damp}$. One observes that at late times, when thermal equilibrium is approached, all fermion and scalar mode temperatures become constant and agree: $T_p^{(f)}(t) = T_p^{(s)}(t) = T_{\rm eq}$. In contrast, there are sizable deviations from the thermal result even for times considerably larger than the characteristic damping time.

Kinetic prethermalization.—In contrast to the rather long thermalization time, prethermalization sets in ex-

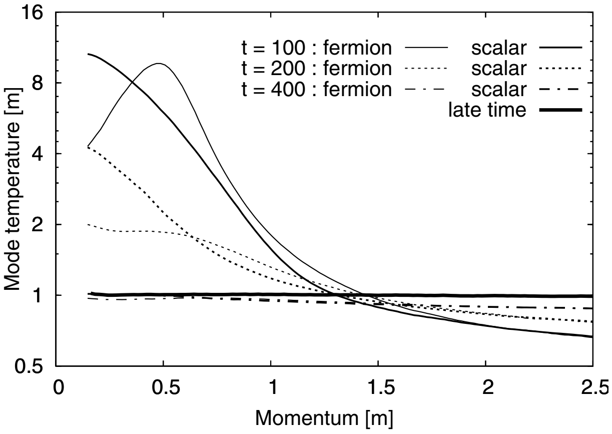

FIG. 2. Fermion and scalar mode temperatures $T_p^{(f,s)}(t)$ as a function of momentum p for various times.

tremely rapidly. In Fig. 3 we show the ratio of pressure over energy density, $w = p/\epsilon$, as a function of time. One observes that an almost time-independent equation of state builds up very early, even though the system is still far from equilibrium. The prethermalization time $t_{\rm pt}$ is here of the order of the characteristic inverse mass scale m^{-1}. This is a typical consequence of the loss of phase information by summing over oscillating functions with a sufficiently dense frequency spectrum. In order to see that this phenomenon is not related to scattering or to the strength of the interaction, we compare with a smaller coupling in the inset and observe good agreement of both curves. The dephasing phenomenon is unrelated to the scattering-driven process of thermalization.

Given an equation of state, the question arises whether there exists a suitable definition of a global kinetic temperature $T_{\rm kin}$. In contrast to a mode quantity such as $T_p(t)$, a temperature measure which averages over all momentum modes may prethermalize. Building on the classical association of temperature with the mean kinetic energy per degree of freedom, we use here a definition based on the total kinetic energy $E_{\rm kin}(t)$:

$$T_{\rm kin}(t) = E_{\rm kin}(t)/c_{\rm eq}. \tag{3}$$

Here the extensive dimensionless proportionality constant $c_{\rm eq} = E_{\rm kin,eq}/T_{\rm eq}$ is given solely in terms of equilibrium quantities [19]. Since the total energy is conserved, the time scale when equipartition is reached (i.e., $E_{\rm kin}/E$ is approximately constant) also corresponds to a time-independent kinetic temperature. The latter equals the equilibrium temperature $T_{\rm eq}$ if $E_{\rm kin}/E$ has reached the thermal value.

The solid line of Fig. 4 shows $T_{\rm kin}(t)$ normalized to the equilibrium temperature (for $T_{\rm eq}/m = 1$). One observes that an almost time-independent kinetic temperature is established after the short time scale $t_{\rm pt} \sim m^{-1}$. The time evolution of bulk quantities such as the ratio of pressure over energy density w, or the kinetic temperature $T_{\rm kin}$, is dominated by a single short time scale. These quantities approximately converge to the thermal equilibrium values already at early times and can be used for an efficient "quasithermal" description in a far-from-equilibrium situation.

Chemical equilibration.—In thermal equilibrium the relative particle numbers of different species are fixed in terms of temperature and particle masses. A system has chemically equilibrated if these ratios are reached, as observed for the hadron yields in heavy-ion collisions [20]. Obviously, the chemical equilibration time $t_{\rm ch}$ depends on the details of the particle number changing interactions in a given model and $t_{\rm ch} \leq t_{\rm eq}$. In our model we can study the ratio between the numbers of fermions and scalars. For this purpose we introduce the chemical temperatures $T_{\rm ch}^{(f,s)}(t)$ by equating the integrated number density of each species,

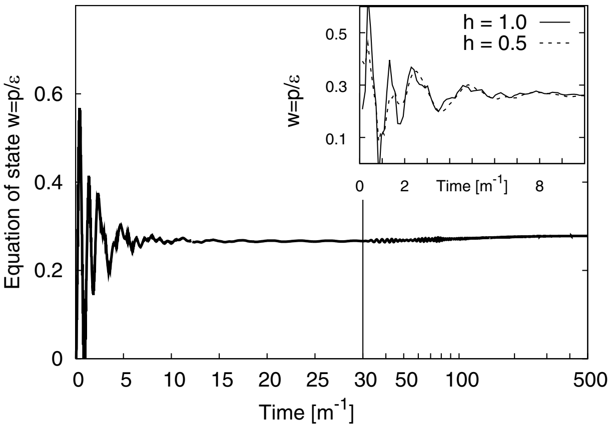

FIG. 3. The ratio of pressure over energy density w as a function of time. The inset shows the early stages for two different couplings and demonstrates that the prethermalization time is independent of the interaction details.

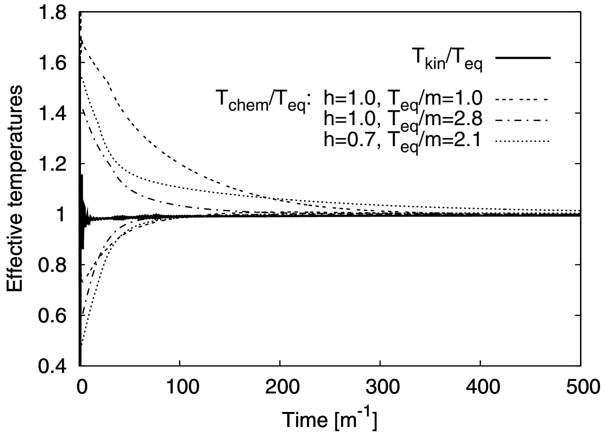

FIG. 4. Chemical temperatures for scalars (upper curves) and fermions (lower curves) for different values of the coupling h and $T_{\rm eq}$. We also show the kinetic temperature $T_{\rm kin}(t)$ (solid line), which prethermalizes on a very short time scale as compared to chemical equilibration.

$n^{(f,s)}(t) = g^{(f,s)} \int d^3p/(2\pi)^3 n_p^{(f,s)}(t)$, with the integrated Bose-Einstein/Fermi-Dirac form of distributions:

$$n(t) \overset{!}{=} \frac{g}{2\pi^2} \int_0^\infty dp\, p^2 \{\exp[\omega_p(t)/T_{\rm ch}(t)] \pm 1\}^{-1}. \qquad (4)$$

Here $g^{(f)} = 8$ counts the number of fermions and $g^{(s)} = 4$ for the scalars.

The time evolution of the ratios $T_{\rm ch}^{(s,f)}(t)/T_{\rm eq}$ is shown in Fig. 4 for different values of the coupling constant h and the equilibrium temperature $T_{\rm eq}$. One observes that chemical equilibration with $T_{\rm ch}^{(s)}(t) = T_{\rm ch}^{(f)}(t)$ does not happen on the prethermalization time scale, in contrast to the behavior of $T_{\rm kin}(t)$. Being bulk quantities, the scalar and fermion chemical temperatures can approach each other rather quickly at first. Subsequently, a slow evolution towards equilibrium sets in. For the late-time chemical equilibration we find for our model $t_{\rm ch} \simeq t_{\rm eq}$. However, the deviation from the thermal result can become relatively small already for times $t \ll t_{\rm eq}$.

Let us finally consider our findings in view of collisions of heavy nuclei and try to estimate the prethermalization time. Actually, $t_{\rm pt}$ is rather independent of the details of the model, such as particle content, values of couplings, etc. It mainly reflects a characteristic frequency of the initial oscillations. If the "temperature" (i.e., average kinetic energy per mode) sets the relevant scale one expects $Tt_{\rm pt} = {\rm const.}$ (For low T the scale is replaced by the mass.) For our model we indeed find $Tt_{\rm pt} \simeq 2$–2.5 [21]. We expect such a relation with a similar constant to hold for the quark-gluon state very soon after the collision [22]. For $T \gtrsim 400$–500 MeV we obtain a very short prethermalization time $t_{\rm pt}$ of somewhat less than 1 fm. This is consistent with very early hydrodynamic behavior [24]. In QCD the equilibrium equation of state shows no strong temperature dependence above the critical temperature $T_{\rm c}$ [25] and can therefore adapt easily as the temperature decreases. After the transition w readjusts only somewhat to the equilibrium value relevant for a hot hadron gas, typically on a time scale of a few fm (for $T \simeq 175$ MeV). The chemical equilibration time $t_{\rm ch}$ depends on the production rate for multistrange hadrons [26]. From $t_{\rm pt} \ll t_{\rm ch}$ and Fig. 4 we conclude that once the chemical temperatures for the various different species are equal the relevant chemical temperature $T_{\rm ch}$ coincides with $T_{\rm kin}$ and defines a universal temperature. Comparison with the critical temperature in equilibrium is therefore meaningful—an approximate equality $T_{\rm ch} \simeq T_{\rm c}$ has been advocated [26]—such that chemical freeze-out can, in principle, be used to measure $T_{\rm c}$.

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[15] With the rescaling $h \to h/N_f$ this corresponds to a nonperturbative expansion of the 2PI effective action to next-to-leading order in N_f [2,6].

[16] The thermal mass m is evaluated in equilibrium. It is found to prethermalize very rapidly. The employed momentum cutoff is $\Lambda/m = 2.86$.

[17] This quantity is directly related to the expectation value of the vector component of the field commutator $\langle[\psi, \bar{\psi}]\rangle$ in Wigner coordinates and fulfills $0 \le n^{(f)}(t; p) \le 1$ [6].

[18] The rate $1/t_{\rm damp}(p)$ is determined by the spectral component of the self-energy [6].

[19] For a relativistic plasma one has $E_{\rm kin}/N = \epsilon/n = \alpha T$. As alternatives, one may consider the weighted average $\bar{T}(t) = \sum n(t; p)T(t; p)/\sum n(t; p)$, where the sum is over all modes, or a definition analogous to Eq. (4).

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[21] We define $t_{\rm pt}$ by $|w(t_{\rm pt}) - w_{\rm eq}|/w_{\rm eq} < 0.2$ for $t > t_{\rm pt}$.

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