Universal Quantum Computation through Control of Spin-Orbit Coupling

D. Stepanenko and N. E. Bonesteel

National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32310, USA

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We propose a method for quantum computation which uses control of spin-orbit coupling in a linear array of single electron quantum dots. Quantum gates are carried out by pulsing the exchange interaction between neighboring electron spins, including the anisotropic corrections due to spin-orbit coupling. Control over these corrections, even if limited, is sufficient for universal quantum computation over qubits encoded into pairs of electron spins. The number of voltage pulses required to carry out either single-qubit rotations or controlled-NOT gates scales as the inverse of a dimensionless measure of the degree of control of spin-orbit coupling.

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Several quantum computation schemes are based on using the spin-$1/2$ degrees of freedom of electrons or certain nuclei as qubits [1–3]. For example, in the proposal of Loss and DiVincenzo [1], qubits are taken to be spins of single electrons trapped in quantum dots. Here we present a method for using spin-orbit coupling in such a system to perform universal quantum computation.

In many spin-based quantum computation schemes two-qubit gates are carried out by switching on and off the exchange interaction between neighboring spins [4,5]. For perfectly isotropic exchange, these two-qubit gates conserve total spin and so have too much symmetry to form a universal set; i.e., they cannot be used to carry out arbitrary unitary transformations on single-spin qubits. A universal set can be realized if single-spin rotations are possible [1], but it is generally believed these will be harder to achieve than two-qubit gates. An attractive alternative is to use an encoding scheme for which isotropic exchange alone is universal [6]. This requires encoding logical qubits into three or more spins [7,8].

Spin-orbit coupling leads to anisotropic corrections to the exchange interaction [9] which, under certain conditions elaborated on below, retains a residual rotational symmetry about a fixed axis. For many purposes these corrections are innocuous. The resulting exchange gates still form a universal set when combined with single-spin rotations [10,11]. And, through a combination of pulse shaping and locally defined spin quantization axes, they can be made effectively isotropic, although in general only to second order in spin-orbit coupling, so that exchange-only encoding can be used [12,13].

The partial reduction in symmetry, from isotropic to axial, can also simplify the requirements for universal quantum computation. In [14] it was shown that the XY interaction is universal for qubits encoded into only two spins, provided there is a third ancillary spin for each qubit. And in [15] it was shown that any axially symmetric anisotropic corrections, when combined with single-spin rotations about an axis perpendicular to the symmetry axis of the exchange, can be used to construct a universal set of gates for unencoded qubits.

In this Letter we propose a new method for quantum computation based on the ability to *control* the spin-orbit induced anisotropic corrections to the exchange interaction in a linear array of GaAs quantum dots. Our proposal requires encoding logical qubits into pairs of neighboring spins, similar to the encoding used in [16–18]. However, unlike these proposals, which require an inhomogeneous Zeeman field in addition to exchange, our proposal employs only the spin-orbit corrected exchange interaction.

Spin-orbit coupling is a relativistic effect which occurs because an electron moving in an electric field experiences a magnetic field which couples to its spin. In solids, the $\mathbf{k}$-dependent spin splitting due to spin-orbit coupling is described by the Hamiltonian $H_{SO} = \Omega(\mathbf{k}) \cdot \mathbf{S}$, where $\mathbf{k}$ and $\mathbf{S}$ are, respectively, crystal momentum and spin. Time-reversal symmetry implies $\Omega(\mathbf{k}) = -\Omega(-\mathbf{k})$; thus $\Omega \neq 0$ only in the absence of inversion symmetry. For a (001) two-dimensional electron gas (2DEG) in GaAs there are two sources of inversion asymmetry contributing to Ω. Taking k_x and k_y to be along the [100] and [010] crystal axes, respectively, the Dresselhaus contribution, $\Omega_D = f_D(-k_x, k_y, 0)$, is due to the bulk inversion asymmetry of the zinc blende structure of GaAs, with coupling f_D inversely proportional to the square of the width of the 2DEG [19], and the Rashba contribution, $\Omega_R = f_R(k_y, -k_x, 0)$, is due to the structural inversion asymmetry of the quantum well forming the 2DEG [20].

In the Hund-Mulliken description of two quantum dots, one Wannier orbital is kept per dot. Let t denote the tunneling amplitude between these orbitals in the absence of spin-orbit coupling. The effect of H_{SO} is to induce a small spin precession during this tunneling. If the dots lie in the (001) plane and are aligned in the [110] direction, the precession axis is fixed to be along the $[1\bar{1}0]$ direction [21]. The precession angle, η, then satisfies

$$\tan\frac{\eta}{2} = s\frac{a_0\omega_0}{\sqrt{2}t}\langle\Psi_1|(k_x + k_y)|\Psi_2\rangle, \tag{1}$$

where Ψ_i is the Wannier state associated with dot i, and

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$$s = \frac{f_D - f_R}{a_0 \omega_0} \tag{2}$$

is a dimensionless measure of the strength of spin-orbit coupling. Here a_0 and ω_0 are, respectively, the linear size and level spacing of a single isolated dot.

If the spin precession axis is fixed during gate operation, and the z axis in spin space is chosen to be parallel to this axis, exchange gates in the presence of spin-orbit coupling will have the form [11]

$$U_{12}(\lambda; \alpha, \beta, \gamma) = e^{-i\lambda H}, \tag{3}$$

where

$$\begin{aligned} H = \mathbf{S}_1 \cdot \mathbf{S}_2 + \frac{\alpha}{2}(S_1^z - S_2^z) + \beta(S_1^x S_2^y - S_1^y S_2^x) \\ + \gamma(S_1^x S_2^x + S_1^y S_2^y) - \tfrac{1}{4}. \end{aligned} \tag{4}$$

Here λ is the integrated strength of the dominant isotropic part of the interaction, and the parameters α, β, and γ characterize deviations from perfect isotropy. The constant $-1/4$ in H corresponds to a particular choice for the overall phase of U which will be convenient in what follows. For small s, $\alpha = C_\alpha s$, $\beta = C_\beta s$, and $\gamma = C_\gamma s^2$ [11]. C_β and C_γ are both of order 1 and depend on the shape and duration of the voltage pulse, though they cannot in general be set to 0. For a generic pulse, C_α is also of order 1 but, because α is odd under time reversal, it can be set to 0 by time-symmetric pulsing [12].

We envision two methods for controlling these anisotropic corrections. One is to control the width and shape of the potential confining the electrons to the 2DEG, thus controlling f_D and f_R, and hence s. For $f_D = f_R$ [22], s can even be set to zero. The other is to control the coefficients C_α, C_β, and C_γ by pulse shaping, as described above (see also [12]). Using these methods, it should be possible to achieve a *continuous range* of gates of the form (3), corresponding to small values of the parameter s. To ensure approximate axial symmetry, we assume a linear array of (001) quantum dots aligned along the [110] direction, as shown in Fig. 1. Note that corrections beyond Hund-Mulliken (i.e., involving more than one orbital per dot) will lead to deviations from perfect axial symmetry and will be a source of error. Here we assume these corrections are small enough to be ignored.

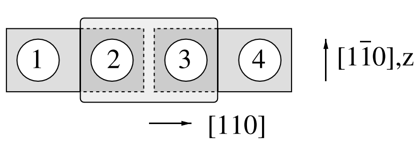

FIG. 1. Four quantum dots forming two neighboring logical qubits, 12 and 34. The dots lie in the (001) plane and are aligned along the [110] direction. The spin-orbit induced spin precession axis is parallel to the [1$\bar{1}$0] direction. Exchange gates between spins within a logical qubit are used for single-qubit rotations. Two-qubit gates are carried out using exchange gates acting on spins 2 and 3.

Because of axial symmetry, the total S_z quantum number of this array will be conserved. It follows that the gates (3) cannot form a universal set if single spins are chosen to represent qubits. We therefore adopt the two-spin encoding scheme of [16–18]. To describe this encoding, we associate a pseudospin space with every nearest-neighbor pair of spins i and $i+1$ spanned by the states

$$|S\rangle_{i,i+1} = \frac{1}{\sqrt{2}}(|\uparrow_i\downarrow_{i+1}\rangle - |\downarrow_i\uparrow_{i+1}\rangle), \tag{5}$$

$$|T_0\rangle_{i,i+1} = \frac{1}{\sqrt{2}}(|\uparrow_i\downarrow_{i+1}\rangle + |\downarrow_i\uparrow_{i+1}\rangle), \tag{6}$$

where $|S\rangle_{i,i+1}$ is pseudospin up and $|T_0\rangle_{i,i+1}$ is pseudospin down. The Hilbert space orthogonal to this pseudospin space is then spanned by the states $|T_+\rangle_{i,i+1} = |\uparrow_i\uparrow_{i+1}\rangle$ and $|T_-\rangle_{i,i+1} = |\downarrow_i\downarrow_{i+1}\rangle$. Given our phase convention, the gates (3) leave this space invariant,

$$U_{i,i+1}(\lambda, \boldsymbol{\phi})|T_\pm\rangle_{i,i+1} = |T_\pm\rangle_{i,i+1}, \tag{7}$$

and so are entirely determined by their action on the pseudospin space,

$$U_{i,i+1}(\lambda, \boldsymbol{\phi}) = e^{i\lambda/2} e^{-i\boldsymbol{\phi}\cdot\boldsymbol{\sigma}^{(i,i+1)}/2}. \tag{8}$$

Here $\boldsymbol{\phi} = \lambda(\alpha, \beta, \gamma + 1)$ and the components of $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices, with the superscript $(i, i+1)$ indicating that they act on the pseudospin space associated with spins i and $i+1$. These gates then correspond to pseudospin rotations through the angle

$$\phi = \lambda(1 + 2\gamma + \alpha^2 + \beta^2 + \gamma^2)^{1/2} = \lambda + O(s^2), \tag{9}$$

about an axis parallel to $\boldsymbol{\phi}$.

In what follows we assume time-symmetric pulsing, so that $\alpha = 0$ for all gates. The available pseudospin rotation axes will then lie in the yz plane. Allowing nonzero α through time-asymmetric pulsing does not appreciably simplify any of our constructions. Given the ability to control the remaining anisotropic terms β and γ, either through direct control of s, or through pulse shaping, there will be a continuous range of available rotation axes. For a given rotation angle, ϕ, these axes will sweep out a wedge shape in the yz plane as shown in Fig. 2. The degree of control of spin-orbit coupling is then characterized by the angular size of this wedge, which we denote θ_m. We expect that θ_m will depend weakly on ϕ and will be on the order of the largest possible value of $|s|$. Note that the wedge of allowed rotation axes need not include the z axis, corresponding to $s = 0$, although as noted above it may be possible to achieve this through cancellation of the Dresselhaus and Rashba contributions.

For logical qubits encoded into the pseudospin spaces of dots i and $i+1$, with i odd, and computational basis states $|0_L\rangle_{i,i+1} = |S\rangle_{i,i+1}$, and $|1_L\rangle_{i,i+1} = |T_0\rangle_{i,i+1}$ (see Fig. 1), we now show how pseudospin rotations can be used to perform single-qubit rotations and controlled-NOT (CNOT) gates, thus providing a universal set of quantum gates [23].

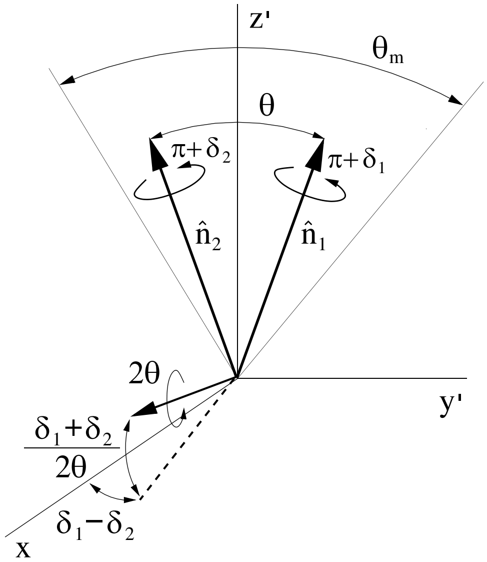

FIG. 2. Rotation axes in the pseudospin space of two neighboring spins. The wedge lying in the plane perpendicular to x and sweeping out the angle θ_m contains rotation axes which can be achieved using time-symmetric pulses and control of spin-orbit coupling. Successive π rotations about $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, with $\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = \cos\theta$, result in a 2θ rotation about the x axis. The effect of errors in the rotation angles, δ_1 and δ_2, on the net rotation axis is also shown. Here $\hat{z}' \parallel (\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2)$ and $\hat{y}' = \hat{z}' \times \hat{x}$.

Consider an arbitrary rotation about the x axis. This operation can be performed by a sequence of π rotations about available axes lying in the wedge. Figure 2 shows two such axes, $\mathbf{n}_1$ and $\mathbf{n}_2$, making an angle $\theta \leq \theta_m$. A π rotation about $\mathbf{n}_1$ followed by a π rotation about $\mathbf{n}_2$ then results in a 2θ rotation about the x axis. The sense of this rotation can be reversed by reversing the order of the π rotations. Since a continuous range of axes within the wedge is available, a rotation about the x axis through an arbitrary angle Θ can be carried out by an even number, $2[\Theta/(2\theta_m)] + 2$, of π rotations, where $[x]$ denotes the greatest integer function of x. The standard Euler construction can then be used to generate arbitrary single-qubit rotations, with the number of pulses required growing as $1/\theta_m$ as θ_m goes to zero.

As θ_m is reduced, this construction also becomes increasingly sensitive to errors. To see this, let the rotation angles about $\mathbf{n}_{1(2)}$ be $\pi + \delta_{1(2)}$, where $\delta_{1(2)}$ are errors. If we take the z' axis to be parallel to $\mathbf{n_1} + \mathbf{n_2}$ and the y' axis parallel to $\hat{z}' \times \hat{x}$ then the composition of these two rotations will yield an overall $2\theta + O(\delta^2/\theta)$ rotation about an axis deviating from the $\hat{x}$ axis by an angle $\delta_1 - \delta_2$ in the y' direction and $(\delta_1 + \delta_2)/2\theta$ in z' direction (see Fig. 2). Thus, the larger θ_m is, the more robust this construction is against errors.

Now consider the two logical qubits shown in Fig. 1. A two-qubit gate between the 12 qubit and the 34 qubit can be carried out by a sequence of pulses acting on spins 2 and 3. Because the pseudospin space of spins 2 and 3 does not correspond to a logical qubit, rotations in this space will, in general, mix in noncomputational states resulting in leakage errors. To avoid such errors, the net unitary transformation must be diagonal in the $\{\uparrow_1\downarrow_2\uparrow_3\downarrow_4, \uparrow_1\downarrow_2\downarrow_3\uparrow_4, \downarrow_1\uparrow_2\uparrow_3\downarrow_4, \downarrow_1\uparrow_2\downarrow_3\uparrow_4\}$ basis of the four spins. The most general unitary operator of the form (8) for which this is the case consists of a rotation about the x axis in pseudospin space. It follows that the net gate must be of the form

$$U_{23}(\Lambda, \Phi) = \prod_k U_{23}(\lambda_k; \boldsymbol{\phi}_k) = e^{i(\Lambda/2)} e^{-i(\Phi/2)\sigma_x^{(2,3)}}, \quad (10)$$

where $\Lambda = \sum_k \lambda_k$ is the net phase and Φ is the rotation angle about the x axis produced by the sequence of rotations $\{\boldsymbol{\phi}_k\}$. Note that both Λ and Φ are defined modulo 4π.

The gate (10) can be expressed in terms of operators acting on the logical qubits as follows:

$$U_{23}(\Lambda, \Phi) = e^{i(\Lambda/4)} e^{i(\Lambda/4)\sigma_x^{(1,2)}\sigma_x^{(3,4)}} e^{i(\Phi/4)\sigma_x^{(1,2)}} e^{i(\Phi/4)\sigma_x^{(3,4)}}. \quad (11)$$

By casting this gate in its canonical form [24], it can be shown to be equivalent to a CNOT gate, up to single-qubit rotations, if and only if

$$\Lambda = \sum_k \lambda_k = (2n + 1)\pi. \quad (12)$$

Below we outline two procedures for simultaneously satisfying (10) and (12).

For the first procedure, let $R_x(\pi)$ be a π rotation about the x axis. Using the single-qubit rotation scheme described above, this rotation can be performed through a sequence of $2n = 2[\pi/(2\theta_m)] + 2$ rotations about available axes. If $A(\phi)$ is then a ϕ rotation about a particular available axis lying in the yz plane, the sequence of rotations $A(\phi)R_x(\pi)A(\phi)$ will have the form (10) with $\Phi = (2n + 1)\pi$ regardless of the value of ϕ. According to (9) the contribution of $R_x(\pi)$ to the total phase Λ will then be $2n\pi + \mu$, where $\mu \sim O(s^2/\theta_m) \sim O(s)$. To satisfy (12) we therefore require $\phi = \pi/2 + O(s)$, where the $O(s)$ adjustment must be chosen so that $\lambda = \pi/2 - \mu/2$ for $A(\phi)$ and thus $\Lambda = (2n + 1)\pi$. This procedure is similar to those proposed in the two-spin encoding schemes of [15–18]. The main difference is that in these constructions the R_x rotation is generated by an inhomogeneous Zeeman field, whereas in ours it is generated entirely by a sequence of exchange gates corresponding to π rotations in the wedge of available axes. Again, as θ_m goes to zero, the number of required pulses scales as $1/\theta_m$ and the construction becomes increasingly sensitive to errors.

The second procedure requires more pulses in the limit of small θ_m but is simpler and less susceptible to error. The idea is to perform a sequence of 2π pseudospin rotations about any available axis or axes and use the spin-orbit induced mismatch between ϕ and λ to accrue the extra π phase required to satisfy (12). The resulting gate will then have the form (10) with $\Phi = 2n\pi$, where n

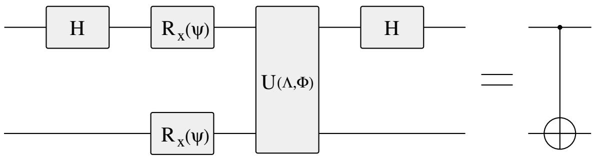

FIG. 3. Proposed CNOT construction. Each line corresponds to a logical qubit. $U(\Lambda, \Phi)$ is defined in (10) with $\Lambda = (2n + 1)\pi$. The value of Φ depends on the procedure used to carry out the CNOT. $H = (\sigma_x + \sigma_z)/\sqrt{2}$ is a Hadamard gate and $R_x(\psi)$ is a single-qubit rotation about the x axis through an angle ψ equal modulo 2π to $(\Phi + \Lambda)/2$.

is the number of 2π rotations. According to (9), for the ith rotation the corresponding phase factor will be $\lambda_i = 2\pi + \nu_i$, where $\nu_i \sim O(s^2)$. For a sequence to satisfy the constraint (12) the sum of all phases, and hence $\sum_i \nu_i$, must be an *odd* multiple of π. Given control of spin-orbit coupling, there will be a continuous range of achievable ν values for each 2π rotation, with $\nu_1 < \nu < \nu_2$, where $\nu_1, \nu_2 \sim O(s^2)$. If this range includes 0, then (12) can always be satisfied with $[\pi/\nu_{\max}] + 1$ rotations, where $\nu_{\max} = \max(|\nu_1|, |\nu_2|)$. If this range does not include 0 it will still always be possible to satisfy (12) with, at most, $[\nu_{\max}/(\nu_2 - \nu_1)] + 2 + [\pi/\nu_{\max}]$ rotations.

Regardless of which procedure is used, single-qubit gates acting on logical qubits 12 and 34 are required to complete the CNOT construction. One procedure for doing this is shown in Fig. 3.

Initialization can be performed by switching on the interaction between pairs of spins forming logical qubits and cooling. If s is set to 0 for this initialization, logical qubits will equilibrate to $|0_L\rangle$. If s cannot be set to 0, they will equilibrate to a state which can be rotated to $|0_L\rangle$. Readout can be performed using a modified version of the scheme proposed by Kane [2]. By switching on tunneling between dots forming a logical qubit, and raising the voltage of one dot so that it becomes doubly occupied if and only if the final state is a singlet, the qubit measurement can be converted to a charge measurement which can be performed using a single electron transistor. If the spin-orbit induced spin precession cannot be turned off during this process, it will not correspond to a measurement in the $\{|0_L\rangle, |1_L\rangle\}$ basis, but rather a measurement along a pseudospin axis nearly parallel to z. Again this does not cause any fundamental problems.

To carry out fault tolerant quantum computation, it must be possible to perform 10^5 gates within the spin decoherence time, τ_s [25]. In GaAs quantum dots, with pulse times of 1 ps [4] and $\tau_s \sim 10$ μs [26], we estimate θ_m must be greater than 0.1 to do this. Given estimates of the size of anisotropic exchange in GaAs [9], we believe this is feasible.

To summarize, we propose a method for quantum computation based on controlling the spin-orbit induced anisotropic corrections to the exchange interaction, with the degree of control characterized by the parameter θ_m. For two-spin encoding of logical qubits, single-qubit rotations and CNOT gates can be carried out with the number of pulses for each scaling as $1/\theta_m$ for small θ_m. For this scheme to be useful it is clearly desirable to design a system for which θ_m is as large as possible.

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