# Experimental Confirmation of the Alhassid-Whelan Arc of Regularity

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(Received 21 May 2004; published 20 September 2004)

Evidence is presented to show that a group of nuclei, spanning a range of structures, corresponds to a previously proposed isolated region of regular behavior between vibrational and rotational structures that was never before observed empirically. Nuclei predicted to show such regular spectra correspond to Hamiltonian parameters that lie amidst those giving more chaotic spectra. We identify a key observable that has a one-to-one correspondence to this arc of regularity and which therefore provides both an empirical signature for it and a clue to its underlying nature.

DOI: 10.1103/PhysRevLett.93.132501 PACS numbers: 21.60.Fw, 21.10.Re, 27.70.+q

The existence of regular (integrable) and chaotic behavior in many-body systems forms a major theme in many branches of physics. Atomic nuclei provide an important testing ground for such behavior because of the variety of structures they exhibit and the dependence of these structures on the number of constituent nucleons. The interacting boson approximation (IBA) [1], which describes collective nuclear excitations as an $N_B$ $s, d$ boson problem, where $N_B$ is the number of bosons (pairs of valence fermions) of $s$ ($L = 0$) and $d$ ($L = 2$) type, forms a versatile model to address this question that has the appeal of being applicable to real systems, i.e., atomic nuclei.

A decade ago Alhassid and Whelan carried out an important study of the chaotic behavior in nuclei using a simple 2-parameter IBA-1 Hamiltonian that spanned the entire range of structures from vibrator to axial rotor to $\gamma$-soft rotor, including all intermediate cases [2]. As expected, they found that the level spectra are regular at and near the three dynamical symmetries U(5), SU(3), and O(6) of the model and that chaotic behavior develops as one moves away from the symmetries. There were two exceptions to this behavior. The first happens between U(5) and O(6) and is connected to the conserved O(5) symmetry [3]. The most fascinating result, however, was that there is a region of nearly regular behavior connecting SU(3) to U(5) centered on an arc in the triangle (see Fig. 1). This arc of regularity amidst chaos could not be explained by a common subgroup. Apparently, along this trajectory there is some hidden symmetry, perhaps corresponding to the emergence of an unidentified conserved or partially conserved quantum number.

At the time, unfortunately, there were no identified nuclei corresponding to this new regular region and this finding therefore remained merely a fascinating theoretical curiosity that was thoroughly studied and confirmed on theoretical ground [4–8] but was not understood in detail nor explored empirically. It is the purpose of this Letter to present evidence that certain atomic nuclei in fact do exhibit this nearly regular behavior, to identify a key observable that uniquely characterizes this region of regularity, and to discuss other observables that identify where along the arc of regularity a given nucleus lies.

Alhassid and Whelan found the quasiregular region using the simple Hamiltonian:

$$\hat{H}(N, \eta, \chi) = c\left[\eta \hat{n}_d + \frac{\eta - 1}{N_B}\hat{Q}_\chi \cdot \hat{Q}_\chi\right], \qquad (1)$$

where $\hat{n}_d = d^\dagger \cdot \tilde{d}$ is the $d$-boson number operator and $\hat{Q}_\chi = [s^\dagger \tilde{d} + d^\dagger s]^{(2)} + \chi[d^\dagger \times \tilde{d}]^{(2)}$ the quadrupole operator. $N_B$ in the denominator stands for the total number of bosons (integral of motion) and ensures a convenient scaling. Control parameters $\eta$ and $\chi$ vary within the range $\eta \in [0, 1]$ and $\chi \in [-\sqrt{7}/2, 0]$. Finally, $c$ is introduced here as a scaling factor needed for comparison with experimental data. The parameter space can be represented by the standard Casten triangle [9] (see Fig. 1)

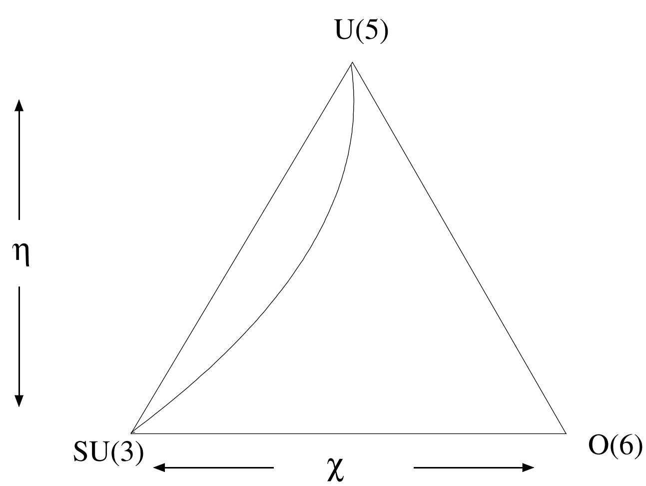


FIG. 1. Locus of the most regular part inside the Casten triangle. Also shown are the three dynamical symmetries at the vertices, and the $\chi$ and $\eta$ coordinates of the triangle.

  

whose $\eta = 1$ vertex corresponds to the U(5) dynamical symmetry (spherical shape), while the dynamical symmetries SU(3) (prolate rotor), O(6) ($\gamma$-soft) are located on the $\eta = 0$ side: SU(3) at the $\chi = -\sqrt{7}/2$ vertex and O(6) at $\chi = 0$. The nearly regular region found in Ref. [2] goes from U(5) to SU(3) through the inner region of the triangle and can be parametrized by the relation [8]:

$$\chi = \frac{\sqrt{7} - 1}{2}\eta - \frac{\sqrt{7}}{2}. \qquad (2)$$

In the $(\eta, \chi)$ plane the location of the regular region is independent of the angular momentum $L$, at low to moderate $L$ values [2], and of $N_B$ [8]. Figure 1 shows the location of the nearly regular region in the Casten triangle.

It has often been thought that real nuclei were located on the edges of the Casten triangle. In the last few years, however, starting with studies in shape or phase transition regions, in which equal weight was given to reproducing the properties of low lying $0^+$ states, significantly revised parameters have been identified that markedly improve the fits of the same IBA-1 Hamiltonian to the data on transitional and deformed nuclei. The most recent of these new studies contains an extensive set of IBA-1 fits to nuclei in the rare-earth region [10] and locates these nuclei in the triangle. In that work the Hamiltonian [11]:

$$\hat{H}(N, \zeta, \chi) = a\left[(1 - \zeta)\hat{n}_d - \frac{\zeta}{4N}\hat{Q}_\chi \cdot \hat{Q}_\chi\right], \qquad (3)$$

was used, where $\zeta$ and $\chi$ are parameters and $a$ is a scaling factor of no structural interest in the present context, but needed for comparison with the experimental data. The parametrizations of Eqs. (1) and (3) are related by:

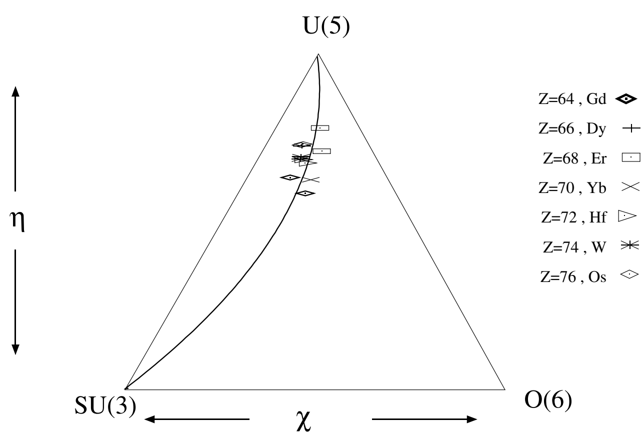


FIG. 2. Location of the fitted parameters (from Ref. [10]) for the 12 selected nuclei. Structure varies highly nonlinearly along the arc: these nuclei vary from near vibrator to near rotor ($R_{4/2} \sim 2.3$ to 3.24).

$$\eta = \frac{4(\zeta - 1)}{3\zeta - 4}; \qquad c = \frac{1}{4}a(4 - 3\zeta). \qquad (4)$$

Using the new parameter values we have searched for nuclei fulfilling Eq. (2), and identified 12 rare-earth nuclei which are located at or near the regular region (see Fig. 2). They are $^{156}$Gd, $^{158}$Gd, $^{156}$Dy, $^{156}$Er, $^{158}$Er, $^{170}$Yb, $^{170}$Hf, $^{172}$Hf, $^{176}$W, $^{178}$W, $^{178}$Os, and $^{180}$Os (several other W, Os nuclei are also quite close to this region).

In Fig. 3 we compare the structure of several of these nuclei with the theoretical description. The overall agreement is good and extends to most of the E2 transition rates [10]. The discrepancies are typical of most IBA calculations using the Hamiltonian of Eq. (3), namely, an expanded scale for energies within the first excited $0^+$ band. The slightly enhanced staggering in the $\gamma$ band is a compromise to fit a wide variety of observables.

The question arises what are the essential distinguishing features of nuclei in or near the regular region. One clue is seen in the middle row in Fig. 3, namely, the close spacing between the $2_2^+$ state, here the $\gamma$ bandhead, and the $0_2^+$ state, forming the bandhead of the first excited $K = 0^+$ band. This can be extended to the other nuclei identified as candidates for the regular region, as seen in Fig. 4, which shows the energy difference between the $2_2^+$ and $0_2^+$ states for all nuclei studied in [10]. Note that this condition sometimes refers to states of different intrinsic

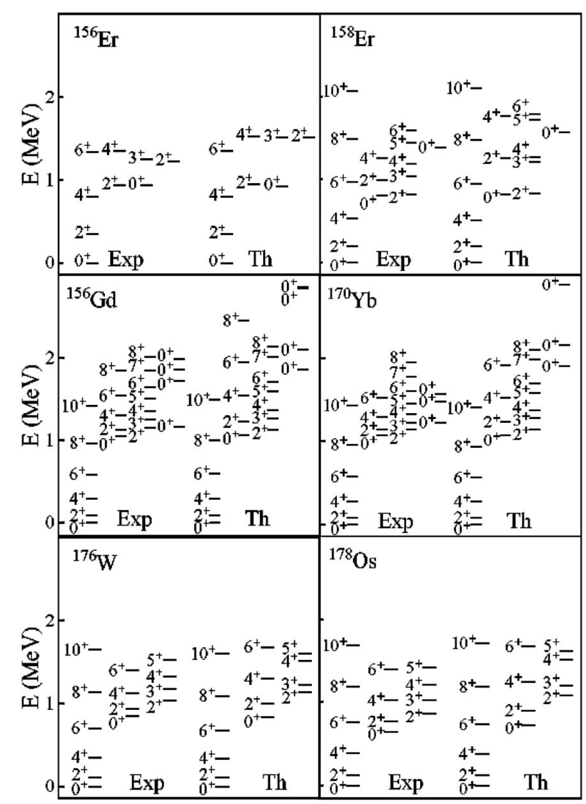


FIG. 3. Comparison between the experimental and theoretical level schemes for $^{156}$Er, $^{158}$Er, $^{156}$Gd, $^{170}$Yb, $^{176}$W, and $^{178}$Os.

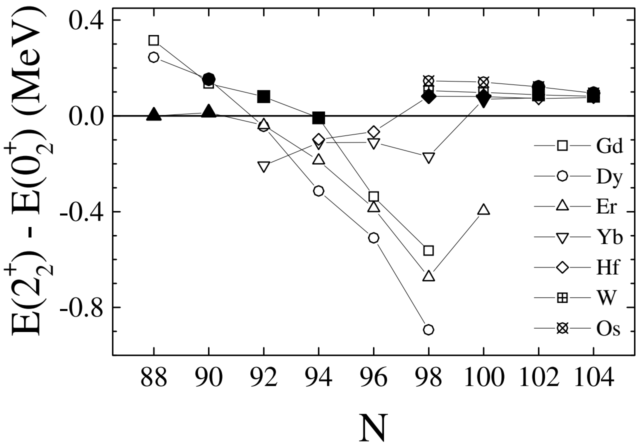


FIG. 4. The energy difference $E(2_2^+)$-$E(0_2^+)$ plotted against neutron number $N$ for the nuclei studied in [10]. Nuclei close to the regular region are shown with solid dots.

structure but in other cases to members of the same rotational band or phonon multiplet.

In the context of the commonly used 2 parameter Hamiltonian of Eq. (1), which accounts quite well for structure throughout the nuclear chart, the near degeneracy of the $2_2^+$ and $0_2^+$ states is, in fact, a unique indicator of the regular region and is its most obvious characteristic. Indeed, as shown in Fig. 5, the locus of the regular region in the triangle and the locus of near degeneracy of the $0_2^+$ and $2_2^+$ states are almost identical. This figure shows the region of the Casten triangle for which the two states are nearly degenerate, $|E(2_2^+) - E(0_2^+)|/E(2_2^+) \leq 0.025$. One observes a very strong correlation, yielding a clear experimental signature. We note that the regular region does not correspond to an exact degeneracy but to close lying $2_2^+$ and $0_2^+$ states and that, in the region of the triangle where the 12 nuclei are located, this energy difference is slightly positive. Thus, in Fig. 4 , the few nuclei with small negative energy differences between $N = 92$ and 96 are, in fact, to the right of the regular region in Fig. 2.

We stress that, with Eq. (1), no other region of the triangle shows this degeneracy. Therefore, identification of nearly degenerate $0_2^+$ and $2_2^+$ states is tantamount to identifying collective nuclei near the regular region. The coincidence of the regular region and this near degeneracy is remarkable. This kind of degeneracy of states belonging to two different spin classes is, in fact, typical of dynamical symmetries such as U(5) or O(6) and therefore may provide clues as to the nature of the underlying quantum number(s) that may be approximately valid in the regular region.

The 12 nuclei span a range of structures along the regular region (note that structure varies nonlinearly with position within the triangle so that the nuclei in Fig. 2 actually differ significantly). Thus the physical nature of the $2_2^+$ and $0_2^+$ changes along the regular region. For the deformed nuclei one obtains degenerate band-

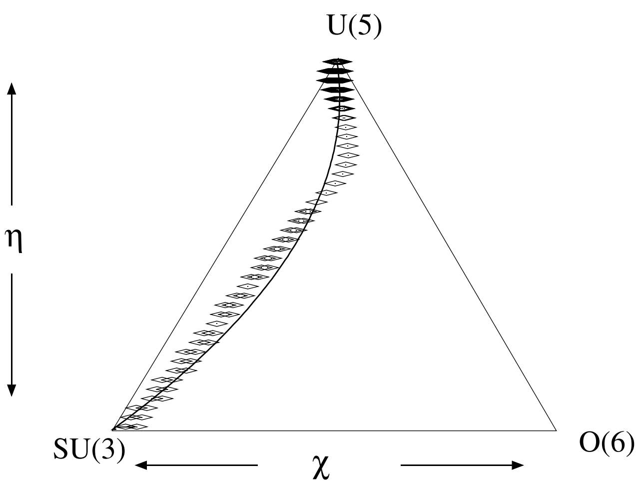


FIG. 5. Comparison between the location of the nearly regular region described by Eq. (2) (solid line) and the part of the parameter space in the Casten triangle for which the second $2^+$ and $0^+$ are almost degenerate (diamonds).

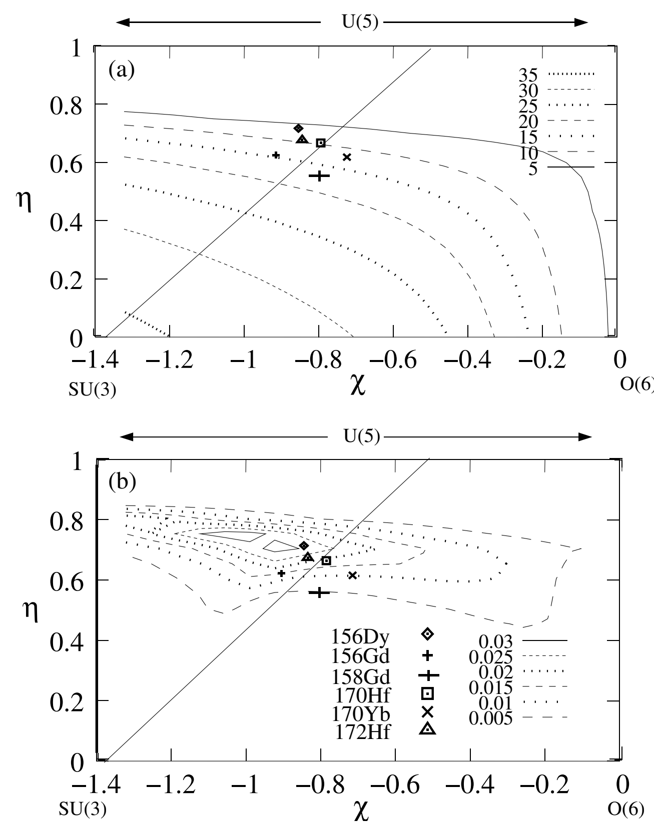


FIG. 6. The theoretical energy ratio $E(0_2^+)/E(2_1^+)$ (a) and $B(E2; 2_2^+ \rightarrow 0_1^+)/B(E2; 2_1^+ \rightarrow 0_1^+)$ ratio (b) for $N_B = 14$. The solid line in each panel shows the locus of the regular region and the symbols indicate the locations of the six most deformed nuclei by their fitted $\eta$ and $\chi$ values.

heads for the $\gamma$ band and the $K = 0_2^+$ band, i.e., for $^{156}$Dy up to $^{172}$Hf (see Fig. 3), while in the less deformed nuclei, i.e., $^{176}$W up to $^{180}$Os, the spacing concerns the two lowest members of the quasi-$K = 0^+$ band. Finally in the nearly spherical nuclei, i.e., $^{156}$Er, the spacing concerns more the two phonon triplet.

We notice in Fig. 5 that a near degeneracy of the $\gamma$ band and the $K = 0_2^+$ band is also characteristic of SU(3). In fact, this is a well known signature of SU(3). Therefore the question arises as to what experimental features distinguish the nuclei we have been discussing (which lie in the interior of the triangle) from those near SU(3). More generally, while the $2_2^+/0_2^+$ degeneracy marks the regular region, what other observables pinpoint where a nucleus is along the arc of regularity? In fact, the fits of Ref. [10] took account of a variety of data in order to pinpoint the parameters of any given nucleus. Here we will illustrate this with two examples. In Fig. 6(a) the energy ratio $E(0_2^+)/E(2_1^+)$, calculated with the Hamiltonian of Eq. (1), is plotted. The contours correspond to constant values of this ratio and the line sloping up from lower left marks the trajectory of the regular region. In SU(3), this ratio is linear in boson number $N_B$: for boson numbers typical of deformed nuclei ($N_B \sim 14$) it takes on values near 30. The six deformed nuclei that are candidates for the regular region in the deformed $A \sim 150$–170 region are located in Fig. 6(a) using the fit parameters obtained in Ref. [10]. It is observed that their energy ratios range from $\sim$5 to $\sim$13, far from the SU(3) value. Of course, arbitrary SU(3) energy ratios can be obtained by adding an $L.L$ term to Eq. (1). Since this would perhaps be artificial and would produce large discrepancies for other states, it is therefore worthwhile identifying an additional signature.

Figure 6(b) shows the $B(E2)$ ratio, $B(E2; 2_2^+ \rightarrow 0_1^+)/B(E2; 2_1^+ \rightarrow 0_1^+)$, which relates interband to intraband transitions and is therefore generally small. It vanishes rigorously in SU(3) but for the area of the triangle where the locus of the regular region crosses the group of six nuclei, it is on the order of $\sim$0.02 to $\sim$0.03. Given the high collectivity of the rotational $B(E2)$ value in the denominator, such ratios correspond to collective interband transitions, again, in distinction to SU(3). Note that the nucleus $^{156}$Dy is at the same time close to both $X(5)$ [e.g., $E(0_2^+)/E(2_1^+) = 4.9$ compared to 5.67 in $X(5)$ [12]] and close to the regular region.

Finally, the presence of nearly regular behavior can also be observed in another feature of the experimental spectra, namely, in the very small spacing between some higher excited $J^+$ states, which are only a few tens of keV apart [13] (see the highly excited $0^+$ states in the theoretical spectrum of $^{156}$Gd shown in Fig. 3). Although this is only observed in $^{172}$Hf, closely spaced experimental states with the same spin and parity are a physically interesting signature for regularity: they imply little or no mixing and therefore imply nearest neighbor level spacings that can be small, as is characteristic of a Poisson distribution. A similar experimental signature involving clustering of levels was recently observed in odd-$A$ nuclei around $^{195}$Pt revealing the presence of a pseudo-orbital symmetry [14]. It may be that the $J^+$-state clustering points to a new symmetry in the regular region.

To summarize, we have identified atomic nuclei located close to the regular region of the Casten triangle noted by Alhassid and Whelan. A dozen possible candidates, spanning the whole range from well deformed to spherical nuclei, were selected using a recent fit of rare-earth nuclei. A unique correlation was discovered between the locus of near degeneracy of the $2_2^+$ and $0_2^+$ states and the regular region. Finally, experimental signatures pinpointing nuclei along the arc of regularity were identified.

We are grateful to Y. Alhassid for many discussions. This work was supported by the BMBF under Project No. 06K167, the USDOE under Grant No. DE-FG02-91ER40609, the GAČR under Grant No. 202/02/0939, and the DFG Grant No. 436 TSE 17/06/03.

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