All Conjugate-Maximal-Helicity-Violating Amplitudes from Topological Open String Theory in Twistor Space

Radu Roiban[1] and Anastasia Volovich[2]

[1]*Department of Physics, University of California, Santa Barbara, California 93106, USA*
[2]*Kavli Institute for Theoretical Physics, Santa Barbara, California 93106, USA*

(Received 26 April 2004; published 23 September 2004)

It has recently been proposed that the D-instanton expansion of the open topological B model on $\mathbb{P}^{3|4}$ is equivalent to the perturbative expansion of the maximally supersymmetric Yang-Mills theory in four dimensions. In this letter we show how to construct the gauge theory results for all n-point conjugate-maximal-helicity-violating amplitudes by computing the integral over the moduli space of curves of degree $n-3$ in $\mathbb{P}^{3|4}$, providing strong support to the string theory construction.

DOI: 10.1103/PhysRevLett.93.131602 PACS numbers: 11.25.–w, 11.15.–q, 12.38.–t

Introduction.—The construction of efficient computational methods for field theory scattering amplitudes has benefited substantially from string theory input. One example is the Bern-Kosower method [1]. The basis of this technique is the low energy limit of string scattering amplitudes combined with the spinor helicity. This results in enormous calculational simplifications for loop amplitudes by taking advantage of the cancellations among Feynman diagrams manifest in the string amplitudes. Another example is provided by the Kawai-Lewellen-Tye relations [2] among gravity and gauge theory scattering amplitudes. At tree level these relations are a consequence of the factorization of the world sheet theory into chiral conformal field theories.

Recently Witten proposed a new technique for computing scattering amplitudes for $\mathcal{N}=4$ supersymmetric gauge theory, or $\mathcal{N}=4$ super-Yang-Mills (SYM), in four dimensions [3]. In this construction the gauge theory scattering amplitudes are given by the D-instanton expansion [4] of a topological open string theory (B model) whose target space is the twistor space of the physical space. The intuition at the base of this conjecture stems from two sources. The first one is the work of Nair [6], who showed that the maximal-helicity-violating amplitudes (MHV) can be expressed in terms of correlation functions of 2-dimensional fermionic currents. The second one is the fact that in a large number of examples described in [3] the scattering amplitudes are supported on special curves when transformed to the twistor space [7] of Minkowski space.

The proposal was tested in [3] for the case of the MHV amplitude and in [8] for the 5-point conjugate-maximal-helicity-violating ($\overline{\text{MHV}}$) amplitude. While these results may seem trivial from the gauge theory perspective, they are certainly not so from the string theory standpoint. Indeed, the gauge theory relation between MHV and $\overline{\text{MHV}}$ amplitudes is just complex conjugation. As we will briefly review in the next section, the string theory construction does not imply any simple relation between them, since they are given by integrals over moduli spaces of curves of different degrees (namely degree one for MHV and an arbitrarily high degree for the conjugate). Moreover the string theory construction suggests that the amplitudes may receive nonvanishing contributions from disconnected curves connected by the twistor space propagator of the gauge theory fields. Perhaps the most intriguing part of the result of [8] is that the 5-point $\overline{\text{MHV}}$ amplitude, though supported on degree two curves, can be recovered without contributions from two disconnected degree one curves.

In this letter we will show that the proposal [3] successfully recovers all $\overline{\text{MHV}}$ n-point amplitudes. As in the case of the 5-point amplitude (which needs to be analyzed by a method different than the one we will use here), we find the rather surprising result that the connected instantons yield the full amplitude without additional contributions from the disconnected curves.

Gluon amplitudes and topological B model.—The gauge theory results for the $\overline{\text{MHV}}$ amplitudes (amplitudes with $n-2$ negative helicities and two positive helicities) is well known: they are complex conjugate to MHV amplitudes (amplitudes with $n-2$ positive helicities and two negative helicities) [9]. In the spinor helicity notation $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ they are given by the simple expression

$$A_{\overline{\text{MHV}}}(\lambda, \tilde{\lambda}, \eta) = ig^{n-2}(2\pi)^4 \delta^4\Big(\sum_{i=1}^n \tilde{\lambda}_i^{\dot{a}} \lambda_i^a\Big) \times \int d^{4n}\psi e^{i\sum_{i=1}^n \eta_{iA}\psi_i^A} \delta^8\Big(\sum_{i=1}^n \tilde{\lambda}_i^a \psi_i^A\Big) \times \prod_{i=1}^n \frac{1}{[i, i+1]}, \tag{1}$$

where the spinor product $[i, j]$ is defined as $[i, j] \equiv [\tilde{\lambda}_i, \tilde{\lambda}_j] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_i^{\dot{b}}$.

Recently Witten [3] conjectured that tree-level n-particle gauge theory amplitudes are supported on curves of degree $d = q-1$, where q is the number of negative helicity gluons. He also proposed that gluon scattering amplitudes in $\mathcal{N} = 4$ SYM can be computed from the open topological B model on the supertwistor space $\mathbb{P}^{3|4}$ in the presence of D5- and D1-branes. In this construction SYM fields are realized as the excitations of the open strings on a D5-brane and the scattering amplitudes are computed by "integrating out" the D1-brane fields. The introduction of D1-branes breaks most of the isometries of $\mathbb{P}^{3|4}$, and in order to restore them, one has to integrate over all possible configurations of the D1-branes whose genus and degree are determined by the amplitude of interest.

Putting together all the details and parametrizing the moduli space of genus zero and degree d curves in terms of degree d maps from $\mathbb{P}^1$ into $\mathbb{P}^{3|4}$, we find that the master formula for the tree-level contribution to n-gluon scattering from instantons of degree d (relevant when there are $d+1$ negative helicity gluons) is [3,8]

$$B(\lambda, \mu, \psi) = \int \frac{d^{4d+4} a d^{4d+4} \beta d^n \sigma}{\mathrm{vol}[GL(2)]} \prod_{i=1}^{n} \frac{1}{\sigma_i - \sigma_{i+1}} \times \delta^3\left(\frac{z_i^I}{z_i^J} - \frac{P^I(\sigma_i)}{P^J(\sigma_i)}\right) \delta^4\left(\frac{\psi_i^A}{z_i^J} - \frac{G^A(\sigma_i)}{P^J(\sigma_i)}\right), \quad (2)$$

where

$$z^I = P^I(\sigma) = \sum_{k=0}^{d} a_k^I \sigma^k; \qquad \psi^A = G^A(\sigma) = \sum_{k=0}^{d} \beta_k^A \sigma^k; \quad (3)$$

and $z^I = (z^0, z^1, z^2, z^3) = (\lambda^1, \lambda^2, \mu^1, \mu^2)$ are the homogeneous bosonic coordinates on $\mathbb{P}^{3|4}$, ψ^A and $A = 1, 2, 3, 4$ are the fermionic coordinates, σ is the inhomogeneous coordinate on $\mathbb{P}^1$, and $I \neq J$. This equation was used in [3] to recover the MHV amplitudes from $d = 1$ curves and in [8] to recover the $n = 5$ $\overline{\mathrm{MHV}}$ amplitudes from $d = 2$. In the next section we recover n-point $\overline{\mathrm{MHV}}$ amplitudes from (2).

The B-Model Calculation.—In this section we evaluate the Fourier transform $\tilde{B}$ of the B-model amplitude (2) with respect to μ for the case $n = d + 3$, which is relevant to the scattering of $n - 2$ negative and two positive helicity gluons in Yang-Mills (YM) theory. In the following, we will keep d arbitrary in the equations which are independent of the relation between the number of external legs and the degree of the curve.

To simplify the equations, we will analyze separately the bosonic and the fermionic parts of Eq. (2). Choosing the index $J = 0$ in Eq. (2) and Fourier transforming $\mu \to \tilde{\lambda}$, the bosonic part of the amplitude, we find

$$\tilde{B}(\lambda, \tilde{\lambda}) = \int \frac{d^{4(d+1)} a d^n \sigma}{\mathrm{vol(GL(2))}} J_0 \times \left[\prod_{i=1}^{n} \delta\left(\lambda_i^2 - \frac{P^1(\sigma_i)}{P^0(\sigma_i)}\right)\right] \times \exp\left[i \sum_{i=1}^{n} \sum_{k=0}^{d} \frac{\epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} a_k^{\dot{b}} \sigma_i^k}{P^0(\sigma_i)}\right], \quad (4)$$

where

$$J_0 = \prod_{i=1}^{n} \frac{1}{\sigma_i - \sigma_{i+1}}. \quad (5)$$

As in [8] we absorbed λ^1 in λ^2 and $\mu^{\dot{a}}$. We will reinstate it at the end of the calculations through the rescaling $\lambda^2 \to \lambda^2/\lambda^1$ and $\tilde{\lambda}^{\dot{a}} \to \tilde{\lambda}^{\dot{a}} \lambda^1$.

The first step is to fix the symmetries of P^1 and the scale symmetry, which form rank 2 general linear group [GL(2)], by setting the variables a_0^0, σ_1, σ_2, and σ_3 to some arbitrary values at the cost of introducing the Jacobian

$$J_1 = a_0^0 (\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_1). \quad (6)$$

The integral over the $2(d+1) a_k^{\dot{a}}$ moduli is trivial and gives

$$\tilde{B} = \int d^d a d^{d+1} b d^{n-3} \sigma_i J_0 J_1 \times \left[\prod_{i=1}^{n} \delta\left(\lambda_i^2 - \frac{B_i}{A_i}\right)\right] \times \prod_{k=0}^{d} \delta^2\left(\sum_{i=1}^{n} \frac{\tilde{\lambda}_i^{\dot{a}} \sigma_i^k}{A_i}\right). \quad (7)$$

Here we have parametrized the remaining bosonic moduli by a_k (with $a_0 = a_0^0$ unintegrated) and b_k, with

$$A_i = \sum_{k=0}^{d} a_k \sigma_i^k, \qquad B_i = \sum_{k=0}^{d} b_k \sigma_i^k. \quad (8)$$

A simple counting reveals that there are enough δ functions to fix all the integration variables to a discrete set of values. The first goal is to find them. It turns out that, for $d = n - 3$, there is a unique set of σ_i and A_i which satisfies the constraints imposed by the last $2(d+1)$ delta functions. To find them we notice that the corresponding equations are linear in the ratio $r_j = A_1/A_j$ and their solution is

$$\frac{A_1}{A_j} = \frac{[k, 1]}{[k, j]} \frac{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}{(\sigma_j - \sigma_2)(\sigma_j - \sigma_3)} \prod_{\substack{p=4 \\ p \neq k,j}}^{n} \frac{(\sigma_1 - \sigma_p)}{(\sigma_j - \sigma_p)} \quad (9)$$

where $j = 4, \ldots, n$ and k can be chosen to be any number from 4 to n different from j. Since the result should not depend on the choice of k we find that the following equation must be satisfied:

$$\frac{[k_1, 1]}{[k_1, j]} \frac{\sigma_1 - \sigma_{k_2}}{\sigma_j - \sigma_{k_2}} = \frac{[k_2, 1]}{[k_2, j]} \frac{\sigma_1 - \sigma_{k_1}}{\sigma_j - \sigma_{k_1}} \quad (10)$$

for all $j \neq k_1 \neq k_2$. Using the same reasoning for A_2/A_j and A_3/A_j we find two more equations for σ_{k_1}, σ_{k_2}, and σ_j. They are similar to Eq. (10) except that the index 1 is replaced by 2 and 3, respectively. These three equations have a unique solution

$$\sigma_j = \frac{\sigma_2\sigma_3[j,1][2,3] + \sigma_1\sigma_2[j,3][1,2] + \sigma_1\sigma_3[j,2][3,1]}{\sigma_1[j,1][3,2] + \sigma_3[j,3][2,1] + \sigma_2[j,2][1,3]}. \quad (11)$$

The next step is to expose the momentum conservation constraint and cast the remaining constraints into a more useful form. To this end we make use of the remarkable identity

$$\left[\prod_{i=1}^{n}\delta\left(\lambda_i^2-\frac{B_i}{A_i}\right)\right]\prod_{k=0}^{d}\delta^2\left(\sum_{i=1}^{n}\frac{\tilde{\lambda}_i^{\dot{a}}\sigma_i^k}{A_i}\right)=J_2\delta^4(p)\left[\prod_{i=1,i\neq 2,3}^{n}\delta\left(\lambda_i^2-\frac{B_i}{A_i}\right)\right]\times\prod_{k=1}^{d}\delta\left(\sum_{i=1}^{n}\frac{[i,2]\sigma_{i3}\sigma_i^{k-1}}{A_i}\right)\delta\left(\sum_{i=1}^{n}\frac{[i,3]\sigma_{i2}\sigma_i^{k-1}}{A_i}\right), \tag{12}$$

where the new Jacobian is

$$J_2 = A_2A_3[2,3]^{d+1} \quad \text{and} \quad \sigma_{ij}\equiv(\sigma_i-\sigma_j). \tag{13}$$

As promised, the first delta function enforces the momentum conservation $\delta^4(p)=\delta^4(\sum_{i=1}^n\tilde{\lambda}_i^{\dot{a}}\lambda_i^{\dot{a}})$ after restoring λ_i^1 dependence.

The integrals over the $(d+1)b$ moduli can be easily performed, given the fact that they appear linearly in the Eq. (12). The resulting factor is

$$J_3=\frac{\prod_{i=1,i\neq 2,3}^n A_i}{V_{14\cdots n}}, \tag{14}$$

where $V_{14\cdots n}$ is the Vandermonde determinant of $\sigma_1,\sigma_4,\cdots\sigma_n$.

The last step is to perform the integrals over a_i and σ_i. Perhaps the easiest way to do this is to make use of the fact that the arguments of the last delta functions are linear in A_1/A_i. Changing the integration variables from a_i to $r_i=A_1/A_i$ with $i=4,\ldots,n$ introduces the Jacobian

$$J_4=\frac{A_1^{4-n}\prod_{i=4}^n A_i^2}{V_{14\cdots n}} \tag{15}$$

Using the $d=n-3$ delta functions in Eq. (12), the integrals over the ratios $r_i=A_1/A_i$ as well as over the positions of the fermionic currents σ_i yield

$$J_5=\left|\begin{matrix}[2,i]\sigma_{i3}\sigma_i^{k-1} & [2,i]r_i(k\sigma_{i3}+\sigma_3)\sigma_i^{k-2}\\ [3,i]\sigma_{i2}\sigma_i^{k-1} & [3,i]r_i(k\sigma_{i2}+\sigma_2)\sigma_i^{k-2}\end{matrix}\right|^{-1}. \tag{16}$$

Multiplying the rows in the first block by $([1,3]\sigma_{21})/([2,1]\sigma_{13})$ and subtracting them from the rows of the second block while using the Eq. (9) leads, after some algebra, to the last piece of the bosonic integrals

$$J_5=\frac{1}{\sigma_{23}^d V_{4\cdots n}^2\prod_{i=4}^n([2,i][3,i]r_i)}. \tag{17}$$

The contribution of the fermionic moduli can be easily computed with the result

$$F\equiv\int d^{4(d+1)}\beta\prod_{i=1}^n\delta^4\left(\psi_i^A-\sum_{k=0}^d\frac{\beta_k^A\sigma_i^k}{A_i}\right)=\left(\frac{V_{14\cdots n}}{A_1A_4\cdots A_n[2,3]}\right)^4\delta^8\left(\sum_{i=1}^n\tilde{\lambda}_i^{\dot{a}}\psi_i^A\right). \tag{18}$$

Evaluating the result of the bosonic integrals $J_0J_1J_2J_3J_4J_5$ on the solution (11) and multiplying by the result of the integral over the fermionic moduli (18) while restoring the λ_i^1 dependence by rescaling λ_i^2, $\tilde{\lambda}_i^{\dot{a}}$, and ψ^A yields the full B-model amplitude

$$\tilde{B}(\lambda,\tilde{\lambda},\psi)=\frac{\delta^4(\sum_{i=1}^n\tilde{\lambda}_i^{\dot{a}}\lambda_i^a)\delta^8(\sum_{i=1}^n\tilde{\lambda}_i^{\dot{a}}\psi_i^A)}{\prod_{i=1}^n[i,i+1]}, \tag{19}$$

in agreement with (1) after the necessary fermionic Fourier transform.

It is important to point out the difference between the gauge theory and the string theory derivation of the MHV and $\overline{\text{MHV}}$ amplitudes. The string theory proposed in [3] directly reproduces the n-point $\overline{\text{MHV}}$ amplitude, while from the gauge theory standpoint they are obtained by solving certain recurrence relations [9–12]. It may turn out that, in general, the most efficient way of proving the proposal [3] is to derive the gauge theory recursion relations from the string field theory of the topological B model on the twistor space. It is, however, our hope that a generalization of the methods described here will prove to be a useful tool in the explicit computation of gauge theory scattering amplitudes.

We have benefited from helpful discussions with D. Gross, D. Kosower, W. Siegel, E. Witten, and especially M. Spradlin. This work was supported in part by the National Science Foundation under PHY00-98395 (RR) and PHY99-07949 (AV), as well as by the DOE under Grant No. 91ER40618 (RR). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[1] Z. Bern and D. A. Kosower, Phys. Rev. Lett. **66**, 1669 (1991).
[2] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B **269**, 1 (1986).
[3] E. Witten, hep-th/0312171.
[4] The expressions for the tree-level gauge theory amplitudes resulting from this construction were recently reinterpreted in [5] in terms of a more standard string theory where the D-instanton expansion is replaced by the perturbative expansion.
[5] N. Berkovits, hep-th/0402045.
[6] V. P. Nair, Phys. Lett. B **214**, 215 (1988).
[7] R. Penrose, J. Math. Phys. (N.Y.) **8**, 345 (1967); R. Penrose, Int. J. Theor. Phys. **1**, 61 (1968).
[8] R. Roiban, M. Spradlin, and A. Volovich, hep-th/0402016.

[9] S. J. Parke and T. R. Taylor, Phys. Rev. Lett. **56**, 2459 (1986).
[10] M. L. Mangano, S. J. Parke and Z. Xu, Nucl. Phys. B **298**, 653 (1988).
[11] M. L. Mangano and S. J. Parke, Phys. Rep. **200**, 301 (1991).
[12] F. A. Berends and W. T. Giele, Nucl. Phys. B **306**, 759 (1988).