One-Channel Conductor in an Ohmic Environment: Mapping to a Tomonaga-Luttinger Liquid and Full Counting Statistics

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(Received 18 December 2003; published 14 September 2004)

It is shown that a one-channel coherent conductor in an Ohmic environment can be mapped to the impurity problem in a Tomonaga-Luttinger liquid. This allows one to determine nonperturbatively the effect of the environment on $I - V$ curves, and to find an exact relationship between dynamic Coulomb blockade and shot noise. We investigate critically how this relationship compares to recent proposals in the literature. The full counting statistics is determined at zero temperature.

DOI: 10.1103/PhysRevLett.93.126602 PACS numbers: 72.10.–d, 72.70.+m, 73.23.Hk, 73.63.Rt

A mesoscopic conductor embedded in an electrical circuit forms a quantum system violating Ohm's laws. The transmission/reflection processes of electrons through the conductor excite the electromagnetic modes of the environment, rendering the scattering inelastic, and reducing the current at low voltage, an effect called dynamical Coulomb blockade (DCB) [1]. This picture, valid in the limit of a weak conductance, changes in the opposite limit of a good conductance [2]. The description of tunneling via discrete charge states becomes then ill defined, raising the question of whether DCB survives or is completely washed out by quantum fluctuations. It is quite clear that DCB vanishes for a perfectly transmitting conductor. This property is shared by shot noise which results as well from the random current pulses due to tunneling events. Such a similarity was concretized [3] through a challenging relationship between the DCB reduction of the current in a one-channel conductor in series with a weak impedance and the noise without impedance (see Fig. 1). More generally, the DCB variation of the $(n-1)$th cumulant of the current was related to the nth cumulant without environment [4]. The environmental effect on the third cumulant has been the subject of recent intensive experimental and theoretical activity [4,5].

An Ohmic environment could as well simulate the electronic interactions in the coherent conductor [6]. In this view, one can wonder whether a one channel conductor in series with a resistance is equivalent to a one dimensional interacting system, described by the Tomonaga-Luttinger liquid (TLL) model [7]. This is already suggested by the power law behavior at small transmission with an exponent determined by $r = R/R_q$ ($R_q = h/e^2$), the dimensionless environmental resistance, instead of the microscopic interaction parameter [1]. Furthermore, it was shown [8] recently that, at low enough energy, a many-channel conductor in series with a weak resistance $r \ll 1$ behaves as a one-channel conductor with an effective energy dependent transmission $\mathcal{T}(E, r)$ similar to that obtained in a weakly interacting one-dimensional wire in the presence of a backscattering center [9]. In this framework, the variation of the current due to DCB is rather given by the shot noise computed through $\mathcal{T}(E, r)$ instead of the bare transmission $\mathcal{T}$.

In this Letter, we fully extend the analogy to a TLL in order to explore the case of an arbitrary resistance r and a good transmission $\mathcal{T}$. It is shown that there is an entire low energy regime where a one channel conductor embedded in its Ohmic environment behaves exactly like a point scatterer in a TLL liquid [10], with a parameter $K' = 1/(1 + r)$.

While coupling to a phonon bath tends to increase the effective TLL parameter [11], here the resistance will rather induce repulsive interactions, which corresponds to $K' < 1$. This allows to use exact field theory results obtained in the TLL context to propose a novel relationship between the DCB current and shot noise, and more generally between all cumulants.

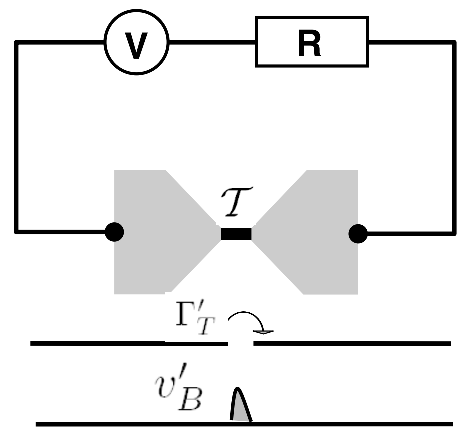

FIG. 1. A one-channel coherent conductor with transmission coefficient $\mathcal{T}$ in series with an impedance $Z(\omega) = R$ for $\omega < \omega_R = 1/RC$. It is mapped to a TLL with parameter $K' = 1/(1 + r)$, where $r = e^2R/h$. The strong (weak) backscattering limit is characterized by a dimensionless amplitude Γ'_T (v'_B).

Consider first a coherent one-channel conductor coupled to its Ohmic environment, and described by the Hamiltonian (for spinless electrons):

$$H = H_1 + H_2 + H_{\rm env} + \Gamma_T \psi_1^\dagger \psi_2 e^{i\varphi(t)+ieVt/\hbar} + \text{H.c.} \quad (1)$$

Here $H_{1,2}$ is the electronic Hamiltonian for the right and left electrodes, $H_{\rm env}$ is a quadratic Hamiltonian describing the electromagnetic modes of the environment, and V is the potential imposed by the voltage generator. The last term couples the phase $\varphi(t)$ across the environmental impedance $Z(\omega)$ to the local electronic fields $\Psi_{1,2}(0)$ at the end points of the left/right electrodes, the momentum dispersion of the tunneling amplitude Γ_T being ignored. We restrict to energies below $\omega_R = 1/RC$, where the conductor capacitance C is included in the total impedance $Z(\omega) = R/(1 + i\omega/\omega_R) \simeq R$ for $\omega < \omega_R$. At zero temperature, the large time behavior of the phase fluctuations becomes [1]

$$\langle \varphi(t)\varphi(0)\rangle - \langle \varphi^2\rangle = -2r \ln(i\omega_R t). \quad (2)$$

The differential dimensionless conductance has been computed to lowest order in the tunneling amplitude [1]

$$G_1 = \frac{h}{e^2}\frac{dI}{dV} \approx \frac{1}{\Gamma(2r+1)}\left(\frac{\Gamma_T}{hv_F}\right)^2\left[\frac{e|V|}{\hbar\omega_R}\right]^{2r}, \quad (3)$$

where Γ is the gamma function. The similarity with the power-law behaviors familiar in TLL is striking. Now it will be shown that it is more than a coincidence.

Since tunneling is punctual, one can use bosonization for the electronic part: Tunneling affects only the s modes, whose dynamics is the same as that of one dimensional leads. One introduces the bosonic field θ with respect to which the electronic Hamiltonian $H_1 + H_2 = H_0^{el}$ in Eq. (1) is quadratic; thus,

$$\langle \theta(t)\theta(0)\rangle - \langle \theta^2\rangle = -\ln(i\omega_F t). \quad (4)$$

The tunneling term becomes $\psi_1^\dagger(0)\psi_2(0) = e^{2i\theta}/2\pi a$, with a a distance cutoff; thus, the total Hamiltonian Eq. (1) reads $H = H_0^{el} + H_{\rm env} + \frac{\Gamma_T}{2\pi a} e^{-2i\theta} e^{i\varphi+ieVt/\hbar} +$ H.c. Since θ and φ commute, and both H_0^{el} and $H_{\rm env}$ are quadratic, respectively, with respect to θ and φ, their sum H_0' is quadratic with respect to the auxiliary field $\theta' = \theta + \varphi/2$, and $H = H_0' + \frac{\Gamma_T'}{2\pi a} e^{2i\theta'(t)+ieVt/\hbar} +$ H.c. Using Eqs. (2) and (4), one gets $\langle \theta'(t)\theta'(0)\rangle - \langle \theta'^2\rangle = -\frac{1}{2K'}\ln(i\omega_F' t)$ up to a constant absorbed in Γ_T', the effective tunneling amplitude. The auxiliary parameter obeys $\frac{1}{K'} \equiv 1 + r$ and $\omega_F' = \min(\omega_R, \omega_F)$. The problem then is formally equivalent to the strong backscattering limit through an impurity in a TLL with an interaction parameter $K' < 1$ and an energy cutoff ω_F'. This equivalence holds not only for the Hamiltonian, but also for all the cumulants of the current. As a quick check, the standard first-order perturbative computation of the average current in the TLL problem [12] yields Eq. (3). In particular, it vanishes at zero voltage, a consequence of the irrelevance of tunneling. Thus, any other neglected scattering process, depending on the realistic setup, could dominate the contribution of tunneling to I at low enough energy, making Eq. (3) nonuniversal.

It is much more useful to think instead of the "dual limit" of weak backscattering with amplitude v_B, thus $\mathcal{T} = [1 + (v_B/hv_F)^2]^{-1}$ close to 1. In the absence of coupling to the environment, the problem is nothing but free electrons in the presence of a potential scatterer, whose locality allows one to use again bosonization. A bosonic field $\Phi(x)$ determines the electronic density through $\rho = -\partial_x\Phi/\pi$; thus, the current $j = e\partial_t\Phi/\pi$. For pedagogical reasons, here we present only arguments at the level of the Euclidian action. It is convenient in this limit to integrate the bulk degrees of freedom and formulate the problem purely in terms of the local field at the impurity $\phi = \Phi(x=0)$ [12]. If τ is the imaginary time and ω_n are the Matsubara frequencies, one has $\hbar S_{el} = \frac{1}{\beta}\sum_{\omega_n}|\omega_n||\phi(\omega_n)|^2 + \frac{v_B}{\pi a}\int_0^\beta d\tau \cos 2\phi(\tau)$. The coupling to the impedance with a fluctuating potential drop $eu_{\rm env} = \hbar\partial_\tau\varphi$ is described by a term $Qu_{\rm env}$, where the transferred charge Q can be identified as $e\phi$. Thus, the action acquires an additional part $\delta S = \int d\tau\, \phi(\tau)(eV/\hbar + \partial_\tau\varphi)/\pi$ with V the applied voltage. Performing a partial integration over the field φ whose corresponding truncated action is [13]: $S_{\rm env} = \sum_{|\omega_n|<\omega_R}|\omega_n||\varphi(\omega_n)|^2/(2Re^2\pi\beta)$ leads to a renormalization of the kinetic term, $|\omega_n| \to |\omega_n|(1+r)$. There is a formal equivalence to one impurity problem in a TLL, this time in the weak backscattering limit and at low energy compared to $\omega_F' = \min(\omega_F, \omega_R)$. Remarkably, one gets the same auxiliary parameter as that found in the tunneling regime, $\frac{1}{K'} \equiv 1 + r$. The auxiliary amplitude v_B', proportional to v_B, will be taken as dimensionless in the following. The advantage of this limit is that the cosine term now defines a relevant perturbation. Thus, the predictions of the field theory are universal as long as v_B' is small enough. The generating Keldysh functional for ϕ turns out to be identical to that in the auxiliary TLL model. Thus, one can exploit known results both for average current [12] and higher cumulants defined by [14]:

$$I_n = \int \langle\langle j(t_1)\cdots j(t_n)\rangle\rangle_c\, dt_1\cdots dt_{n-1}, \quad (5)$$

with c indicating the connected part of the nth symmetrized current correlator. In the following, we will introduce the differential dimensionless cumulants:

$$G_n = \frac{h}{e^{n+1}}\frac{dI_n}{dV}. \quad (6)$$

Let us first discuss the differential conductance G_1 inferred from lowest order perturbation with respect to v_B', in the limits where k_BT/eV is either small or large [12]:

$$G_1 = \frac{h}{e^2}\frac{dI}{dV} = K' - c(K')v_B'^2\left(\frac{\omega}{\omega_F'}\right)^{2(-1+K')}, \qquad (7)$$

where $\hbar\omega = \max(k_BT, eV)$, and $c(K')$ a function of K'. First, observe that, for $v_B' = 0$, one has a purely linear regime with Ohm's law restored: $I = (e^2/h)K'V$, which translates into $V = (R + R_q)I$. This is nothing but the series resistance of a perfect point contact with resistance $R_q = h/e^2$ and the resistor R. Second, a bare amplitude v_B' is modified into an effective larger amplitude $v_B'\omega^{-1+K'}$ which diverges at low ω because $K' < 1$; thus, the above perturbative result is valid above a voltage scale $V_B' \propto \omega_F' v_B'^{1/(1-K')}$. Increasing ω up to ω_F' drives G_1 to its maximum value, $G_{\max} = K' - c(K')v_B'^2$ which is still smaller than the conductance without environment, $G = 1 - c(1)v_B'^2$. Notice that linearity can be maintained only at $k_BT \gg eV \gg eV_B'$, but breaks at $k_BT \ll eV$. On the other hand, decreasing ω below V_B' increases the effective barrier height, and G_1 drops to zero at zero ω. The low-energy behavior of an almost transparent junction coupled to the environment is, thus, qualitatively similar to the one of a very poorly transmitting junction. In this limit, one can do perturbation with respect to a dimensionless tunneling amplitude Γ_T' related to v_B' in a nonuniversal way. Thus, one gets a similar result to Eq. (3) at $eV_B' \gg eV \gg k_BT$, while eV has to be replaced by k_BT if $eV_B' \gg k_BT \gg eV$. All these considerations can be made nonperturbative using [15]. Thus, increasing the bare transparency of the conductor does not wash out DCB but reduces its domain to $V < V_B'$.

Motivated by the recurrence relation between cumulants suggested in previous works with a restriction to $r \ll 1$ [3,4], we now establish a more general relation holding for an arbitrary r, starting by a comparison of G_1 to the (dimensionless) differential noise G_2 [Eqs. (5) and (6)] at $k_BT \ll eV$. Let us stick first to the two perturbative regimes discussed thus far such that the noise is Poissonnian [16]. More precisely, $G_2 \simeq 2G_1$ for $V \ll V_B'$, while $G_2 \simeq 2K'(K' - G_1)$ for $V \gg V_B'$. Together with the expressions of G_1 in Eqs. (3) and (7), respectively at low and high voltages, one can check the relation, for an *arbitrary* r, and for $n = 2$:

$$\frac{\partial G_{n-1}(V, r)}{\partial \log V} = -2rG_n(V, r). \qquad (8)$$

Notice that the left-hand side would vanish at $r = 0$, because G_{n-1} becomes voltage independent in this limit; thus, this quantity is purely related to the presence of the environmental resistance. This relation expresses that the DCB contribution to the conductance is related to the noise in the presence of the environment. It holds not only at leading order in V/V_B' or V_B'/V, but *to any order*. This truly nonperturbative observation was dubbed a "generalized fluctuation dissipation theorem" in [15]. In order to compare it to the recent related works dealing with a small resistance r, we now restrict to K' close to 1. At strictly vanishing r, $K' = 1$, and both the low and high energy series of the exact differential conductance [15] can be trivially resummed to give the transmission probability $G_1 \equiv \mathcal{T} = 1/(1 + v_B'^2) = \Gamma_T'^2/(1 + \Gamma_T'^2)$. To lowest order in r, it is tempting to replace G_n on the right-hand side (rhs) of Eq. (8) by its value at $r = 0$, here $G_2(V, r) \simeq G_2(V, 0) = \mathcal{T}(1 - \mathcal{T})$: Doing this would suggest that the DCB contribution of a small resistance r to the current would be proportional to the shot noise without environment as argued in [3]. But caution is needed, as can be seen already in the two dual limits where the noise is Poissonnian (3) and (7), $G_2 \sim \Gamma_T^2 V^{-2r/(1+r)}$ or $G_2 \sim v_B^2 V^{2r}$; even if $r \ll 1$, G_2 can be replaced by its noninteracting value only if V is not too small. A more quantitative comparison of $G_2(V, r)$, inferred from the exact solution of [15], to its noninteracting value is given by the continuous curve of Fig. 2: The ratio $x = G_2(V, r)/G_2(r = 0)$ is plotted as a function of $\log V$ at $r = 0.05$. x is close to 1 only for intermediate voltages, a manifestation of the breakup of perturbation theory with respect to r. The quality of this agreement depends on the value of $\mathcal{T}$, as will be discussed elsewhere.

The mapping to a TLL at an arbitrary r and the subsequent exact solution can be used as well to shed some light on Ref. [8] which is in the spirit of Ref. [9] where an effective energy-dependent transmission coefficient $\mathcal{T}(E, r)$ is introduced, and argued to satisfy the following renormalization group equation in the limit of small r: $\partial\mathcal{T}(E, r)/\partial \log E = -2r\mathcal{T}(E, r)[1 - \mathcal{T}(E, r)]$. This formula suggests approximating the differential noise on the rhs of Eq. (8), for $r \ll 1$, as $G_2(V, r) \simeq G_1(V, r)[1 - G_1(V, r)]$. However, this is not satisfactory at high voltages, when $G_1(V, r) \to K' = 1/(1 + r)$. Rather, a better approximation is obtained by defining $\mathcal{T}(V, r) = (1 + r)G_1(V, r)$ such that $G_2(V, r) \simeq \mathcal{T}(V, r)[1 - \mathcal{T}(V, r)]$, as shown through their ratio x in Fig. 2 (the

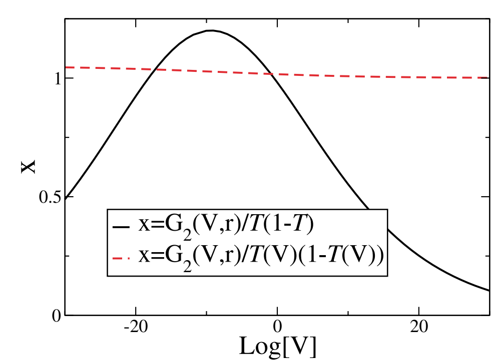

FIG. 2 (color online). Limit of a weak resistance, $r = 0.05$: The ratio x of the exact differential noise $G_2(V, r)$ to that at $r = 0$ (continuous curve), and to a "renormalized" noise given by $\mathcal{T}(V, r)[1 - \mathcal{T}(V, r)]$ (dashed curve) is plotted against $\log V$ where V is implicitly divided by an arbitrary voltage scale. The bare transmission is taken to be $\mathcal{T} = 0.65$, while $\mathcal{T}(V, r) = (h/e^2)(1 + r)dI/dV$.

dashed curve). This approximation is good up to an accuracy of r in intermediate to high voltage regimes.

Remarkably, for an arbitrary resistance r, Eq. (8) can be extended to all cumulants, i.e., to $n > 2$ where G_n in Eqs. (5) and (6) is computed with the environmental resistance in series, i.e., at $K' < 1$[14]. Again, the limit $r \ll 1$ requires care: Replacing G_n on the rhs by its value without environment, $G_n(V, r) \simeq G_n(r = 0)$, yields the prediction of [4], but with a restricted validity domain. Rather, a better fit to [8] is expected if one replaces $G_n(V, r \ll 1)$ by that expressed in the scattering approach through the effective transmission $\mathcal{T}(V, r) = (1 + r)G_1(V, r)$, which needs to be checked.

A study of the various properties for finite temperature requires complex Bethe ansatz calculations, and is identical to the examples carried out in [15]. The case $K' = 1/2$ is particularly simple. This corresponds to a crossover value, $R = R_q$. While the average current and noise have been expressed analytically, it would be interesting to compute the higher cumulants [17].

An interesting extension of these results can be done for a point scatterer in a TLL of parameter K coupled to an Ohmic environment: The auxiliary parameter becomes $\frac{1}{K'} = \frac{1}{K} + r$, which increases the effective interactions by making K smaller; thus, the power law exponent is a combination of effects of the environment and the microscopic interactions. Note, however, that the role of the reservoirs will have to be carefully understood.

In conclusion, we have first seen in this Letter how the coupling to an Ohmic environment induces effective repulsive e-e interactions. While this idea is not entirely new [18], the setup of a well transmitting element coupled to an arbitrary resistance provides a concrete realization, which seems very amenable to experimental study, especially in view of the recent progress in good transmitting atomic contacts [19]. It is particularly exciting to have a potential new way of seeing TLL physics [20], and the dramatic effect of a weak backscattering barrier at low energy. Conceptually, the relationship between TLL and dissipation is not that surprising: Starting inversely from a TLL, an electron can view the surrounding electrons with which it interacts as an effective electromagnetic environment [6]. Thus, a TLL can as well be viewed as the simplest one-channel conductor coupled to a resistor [18]. Beyond its qualitative interest, the mapping has allowed us to make contact with exact field theoretic calculations, yielding the full counting statistics for the current at zero temperature. We have then been able to propose a more general link between the dynamic Coulomb blockade and the shot noise, embodied in the exact relation (8), a "nonequilibrium fluctuation dissipation theorem," whose deep origin remains somewhat mysterious, and which extends to higher cumulants. In particular, the mapping yields the third cumulant at arbitrary environmental resistance, opening the perspective to include the finite temperature, especially feasible at $R = R_q$.

This problematic was motivated by D. Estève, P. Joyez, and C. Urbina: We thank them for their numerous suggestions, and a critical reading of the manuscript. I. S. would like to thank H. Bouchiat, F. Hekking, A. Levy-Yeyati, and Y. Nazarov for interesting discussions, as well as B. Trauzettel for discussions and providing Fig. 2.

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