Kinetic Susceptibility and Transport Theory of Collisional Plasmas

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A system of nonlocal electron transport equations for electrostatic perturbations in (ω, k) space in a high-Z plasma is derived from the Fokker-Planck equation for arbitrary relations between the time, space, and collisionality scales. The closed scheme for obtaining the longitudinal plasma susceptibility $\epsilon(\omega, k)$ in the entire (ω, k) plane is proposed. Regions in the (ω, k) plane have been mapped for problems such as the relaxation of the local temperature enhancement with a time-dependent heat conductivity. The electron dielectric permittivity has been calculated over the entire range of parameters, including the transition region between Vlasov and Fokker-Planck equation solutions.

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Solution of a kinetic equation by reduction to the closed system of hydrodynamic equation is a fundamental problem in statistical description of plasmas that also arises in the kinetic theory of many-body systems in condensed matter or nuclear physics. We present such a solution for an electron plasma which is close to thermodynamical equilibrium and which responds to perturbations at arbitrary length and time scale variations. A complete dielectric function, $\epsilon(\omega, \mathbf{k})$ and nonlocal and nonstationary closure relations for the transport theory based on the solution to the Fokker-Planck (FP) equation are obtained.

The linear theory of nonlocal transport [1] is based on the rigorous procedure of solving the FP equation. The method assumes a local Maxwellian distribution function at the initial time (cf. Ref. [2]) and produces an electron distribution function (EDF) in terms of thermodynamical forces and initial perturbations of density, $\delta n(0)$, and temperature, $\delta T(0)$. Closure relations eliminate initial values $\delta n(0)$, $\delta T(0)$ and provide transport relations for higher moments of the EDF. To date, this scenario has been realized with the assumption of a stationary plasma response. It is clear, however, from the study by Brunner *et al.* [3] that the relaxation of short-wavelength periodic temperature perturbations require a time-dependent thermal conductivity. Our Letter presents the generalization of nonlocal transport theory [1] to time-dependent problems. The published results regarding high-frequency collisional electrostatic perturbations [4–6] and the transport formulation of the collisionless plasmas [7] have already used a similar approach in a limited regime of parameters. From transport relations and specifically from the generalized Ohm's law, one finds the plasma dielectric function as shown for different regimes of parameters in Refs. [1,7–9]. The dielectric properties of the collisional plasma are constantly being examined [10,11] in the transition region between weakly collisional and Vlasov limits. Our results will allow for such studies without involving simplified models of particle collisions.

The electron FP equation is linearized about a Maxwellian distribution function, F_0, with immobile ions. We consider the e-e collision operator in the Landau form and the e-i collision operator in the Lorentz approximation. The perturbation of the EDF is expanded in Legendre polynomials, $f = \sum_{l=0}^{\infty} f_l \times (\omega, k, v) P_l(\mu)$, where $\mu = (\mathbf{v}\cdot\mathbf{k})/(vk)$. Because of the high-$Z$ approximation, the e-e collision term, C_{ee}, contributes only to the equation for the symmetric part of the EDF, f_0. The assumed initial EDF $f(\mathbf{v}, \mathbf{k}, 0)$ is a linearized local Maxwellian with perturbations in density, $\delta n(0)$, and temperature, $\delta T(0)$ [1]. The infinite system of equations for the higher angular harmonics is reduced to an equation for f_1 by applying the summation procedure [1,8,12]. The infinite sum is represented here by the recurrence relation [6] for the renormalized collision frequency ν_l,

$$\nu_l = -i\omega + \frac{l(l+1)}{2}\nu_{ei}(v) + \frac{(l+1)^2}{4(l+1)^2 - 1}\frac{k^2 v^2}{\nu_{l+1}}, \quad (1)$$

where $\nu_{ei}(v) = 4\pi Z n e^4 \Lambda / m^2 v^3$ is the velocity-dependent e-i collision frequency, where e and n are the electron charge and density and Λ is the Coulomb logarithm. The expression for the first angular harmonic of the EDF reads

$$f_1 = -ikv[f_0 - i(eE/kT)F_0]/\nu_1(v), \quad (2)$$

where the isotropic part of the EDF can be expressed in terms of the basic functions ψ^A,

$$f_0 = i\frac{eE}{kT}F_0 + \left(\frac{\delta n(0)}{n} - \omega\frac{eE}{kT}\right)\psi^N F_0 + \frac{3\delta T(0)}{2T}\psi^T F_0, \quad (3)$$

which satisfy the kinetic equations with different source terms, S_A,

$$\left(-i\omega + \frac{k^2 v^2}{3\nu_1}\right)\psi^A = F_0^{-1} C_{ee}[F_0\psi^A] + S_A, \quad (4)$$

where $S_N = 1$ and $S_T = v^2/3v_T^2 - 1$. This equation is solved using standard Laguerre polynomial expansion [1,13]. The initial density and temperature perturbations are eliminated from Eq. (3) by calculating two moments, δn and δT, of this equation. This results in the following expression for the symmetrical part of the EDF, in terms of its hydrodynamical moments

$$f_0 = i\frac{eE}{kT}F_0 + \left(\frac{\delta n}{n} - i\frac{eE}{kT}\right)\frac{J_T^T\psi^N - J_T^N\psi^T}{D_{NT}^{NT}}F_0 + \frac{\delta T}{T}\frac{J_N^N\psi^T - J_N^T\psi^N}{D_{NT}^{NT}}F_0, \quad (5)$$

where $D_{AB}^{CD} = J_A^C J_B^D - J_A^D J_B^C$. We have introduced the velocity moments, J_B^A, of the basic functions $J_B^A = (4\pi/n) \times \int_0^\infty v^2 dv \psi^A F_0 S_B$ which satisfy $J_B^A = J_A^B$ as in the quasistatic case [1].

Two first moments of the kinetic equation for the symmetric part of the EDF, f_0, lead to the hydrodynamic equations

$$\frac{\partial \delta n}{\partial t} - \frac{i}{e}\mathbf{k}\cdot\mathbf{j} = 0, \qquad \frac{\partial \delta T}{\partial t} + \frac{2i}{3n}\mathbf{k}\cdot\left(\mathbf{q} - \frac{T}{e}\mathbf{j}\right) = 0, \quad (6)$$

where $\mathbf{j}$ and $\mathbf{q}$ are the longitudinal components of the current density and the heat flux. From Eqs. (2) and (5) we calculate these moments and derive the nonlocal and nonstationary closure relations in a standard form,

$$\mathbf{j} = \sigma\mathbf{E}^* + \alpha i\mathbf{k}\delta T, \quad \mathbf{q} = -\alpha T\mathbf{E}^* - \chi i\mathbf{k}\delta T. \quad (7)$$

Here, $\mathbf{E}^*$ is the effective electric field naturally appearing both in classical strongly collisional and nonlocal transport theories $\mathbf{E}^* = \mathbf{E} + i\mathbf{k}(T/e)(\delta n/n + \delta T/T)$. The transport coefficients include the electrical conductivity σ, the thermoelectric coefficient α, and the thermoconductivity, χ, and are given by

$$\left\{\frac{\sigma}{e^2}, \frac{\alpha}{e}, \frac{\chi}{T}\right\} = \frac{n}{k^2 T}\left\{\frac{J_T^T}{D_{NT}^{NT}} + i\omega, \quad \frac{J_T^N + J_T^T}{D_{NT}^{NT}} + i\omega, \quad \frac{2J_T^N + J_T^T + J_N^N}{D_{NT}^{NT}} + i\frac{5}{2}\omega\right\}. \quad (8)$$

The transport relation for the electron heat flux is often written in terms of the current and temperature gradients, $\mathbf{q} = -(\alpha T/\sigma)\mathbf{j} - \kappa i\mathbf{k}\delta T$, by using the heat conductivity coefficient $\kappa = \chi - (\alpha^2/\sigma)T$. The transport relations [Eqs. (6)–(8),] are fully equivalent to the linearized FP equation. An electric field, $\mathbf{E}$, can be defined as an ambipolar field ($\mathbf{j} = 0$) by the Poisson equation (cf. [6]) or simply as a prescribed external field. According to Eq. (8), the transport coefficients depend on three dimensionless parameters $k\lambda_{ei}$, ω/ν_{ei}, where $\nu_{ei} = \sqrt{2/9\pi}\nu_{ei}(v_{Te})$ is the standard e-i collision frequency, and Z. We illustrate in Fig. 1 variations of complex transport coefficients σ and χ with frequency for a given $k\lambda_{ei}$.

In the classical strongly collisional limit, $k\lambda_{ei} < \{0.06/\sqrt{Z}, 0.1\omega/\nu_{ei}\}$, the two Laguerre polynomial approximations can be used to solve Eq. (4) and derive the following [13]:

$$\left.\frac{\sigma}{e^2}\right|_{l=0}, \left.\frac{\alpha}{e}\right|_{l=1}, \left.\frac{\chi}{T}\right|_{l=2} = \frac{2^{1-l}}{9\pi}\frac{n}{m}\int_0^\infty dx \exp(-x^2/2) \times x^7(x^2-5)^l/[\nu_{ei} - i\sqrt{2/9\pi}\omega x^3]. \quad (9)$$

In the quasistatic limit, $\omega < \nu_{ei}$, we recover the results of Spitzer-Härm (SH) theory [14],

$$\left\{\frac{\sigma}{e^2}, \frac{\alpha}{e}, \frac{\chi}{T}\right\} = \frac{n}{m\nu_{ei}}\left\{\frac{32}{3\pi} + \frac{70i\omega}{\pi\nu_{ei}}, \quad \frac{16}{\pi} + \frac{210i\omega}{\pi\nu_{ei}}, \quad \frac{200}{3\pi} + \frac{1015i\omega}{\pi\nu_{ei}}\right\}. \quad (10)$$

According to Eq. (10), the imaginary parts of the transport coefficients are small and increase with ω. In the opposite high-frequency limit, $\omega \gg \nu_{ei}$, the transport coefficients

$$\left\{\frac{\sigma}{e^2}, \frac{\alpha}{e}, \frac{\chi}{T}\right\} = \frac{in}{m\omega}\left\{1 - i\frac{\nu_{ei}}{\omega}, \quad \frac{5}{2}\left(\frac{\pi^8\nu_{ei}^5}{36\omega^5}\right)^{1/6} + i\frac{3\nu_{ei}}{2\omega}, \quad \frac{5}{2} - i\frac{13\nu_{ei}}{4\omega}\right\} \quad (11)$$

are almost entirely imaginary with small real corrections.

Deviations from the classical SH results at $k\lambda_{ei} \geq 0.06/\sqrt{Z}$ [1] (cf. Figure 2) correspond to the well-known transition to the nonlocal thermal transport regime. Nonlocal transport is well understood in the quasistationary limit, $\omega \ll \nu_{ee}, \nu_{ei}(k\lambda_{ei})^{4/7}/Z^{5/7}$ [1,9]. From this

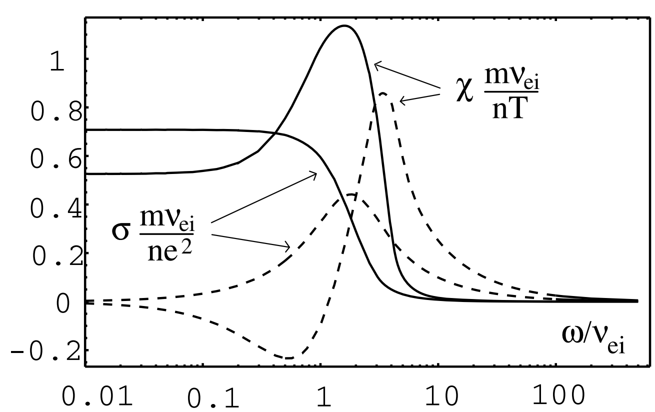

FIG. 1. The dependencies of real (solid lines) and imaginary (dashed lines) parts of the transport coefficients σ and χ on ω/ν_{ei} for $Z = 10$ and $k\lambda_{ei} = 1$.

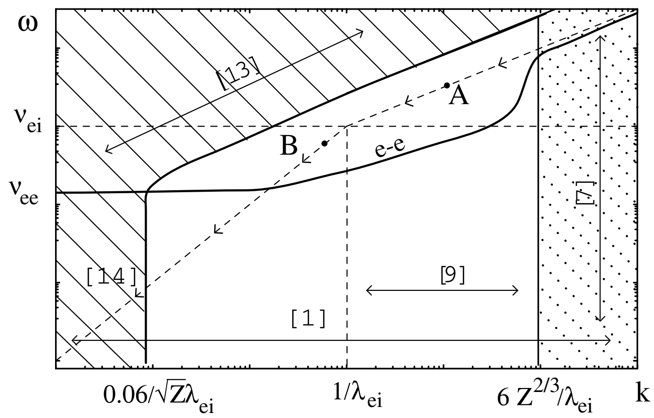

FIG. 2. The parametric plane for the electron transport coefficients. Numbers in square brackets refer to the list of references.

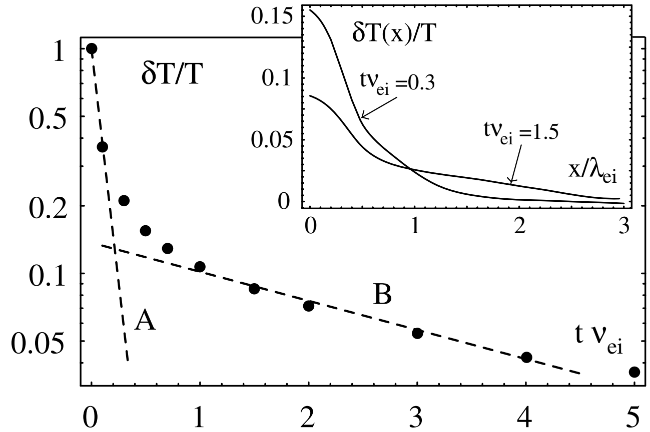

FIG. 3. Time evolution of the temperature in the center of a hot spot between points A ($k\lambda_{ei} \approx 10$) and B ($k\lambda_{ei} < 1$) along the dashed curve in Fig. 2. Characteristic k values correspond to changing widths of temperature profiles as shown in the inset.

region of Fig. 2 moving along vertical lines of increasing frequency, one reaches the local in space, frequency-dependent transport domain [13,14] marked as the dashed region. Between this classical transport domain and the line *e-e* lies the frequency dependent nonlocal electron transport region, addressed in this Letter. The *e-e* boundary, which separates the quasistationary regime (below) from the time-dependent regime (above), is defined for small k values by a $\max\{\nu_{ee}, \nu_{ei}(k\lambda_{ei})^{4/7}/Z^{5/7}\}$ condition. At larger k values, this transition line is obtained by direct comparison of transport coefficients which have been calculated with and without *e-e* collisions. Of course, none of the line boundaries drawn in Fig. 2 indicate sharp interfaces between different transport approximations. They represent transitional regions in the ω, k plane that extend for up to the order of magnitude variation in these parameters.

Relatively simple expressions for transport coefficients are found in the (ω, k) domain where one can neglect C_{ee} in Eq. (4) [5], i.e., for $\omega \gg \nu_{ee}, \nu_{ei}(k\lambda_{ei})^{4/7}/Z^{5/7}$ (above the *e-e* curve in Fig. 2). In this case, the basic functions are approximated by $\Psi^{N,T} = 3\nu_1 S_{N,T}/(k^2 v^2 - 3i\omega\nu_1)$, leading to the following expressions for the moments

$$J_N^N\Big|_{l=0}, J_N^T\Big|_{l=1}, J_T^T\Big|_{l=2} = 3^{1-l}\sqrt{\frac{2}{\pi}}\int_0^\infty dx \exp(-x^2/2) \times (x^2-3)^l/[k^2 v_T^2/\nu_1(x) - 3i\omega/x^2], \qquad (12)$$

($x = v/v_T$) that allow calculations of all transport coefficients in the explicit form. Note that the *e-e* curve in Fig. 2 turns up sharply at $k\lambda_{ei} \sim 6Z^{2/3}$ approaching the line $\omega = kv_T$, which separates the quasistationary and time-dependent limits in the collisionless plasmas, $k\lambda_{ei} > 6Z^{2/3}$ (dotted domain in Fig. 2). This sharp increase is due to large contributions of high order angular harmonics of the EDF for large gradients, $k\lambda_{ei} \sim 6Z^{2/3}$. In the opposite limit of $k\lambda_{ei} \ll 6Z^{2/3}$, nonlocal transport theory can be developed by using a diffusive approach, where only f_0 and f_1 are taken into account [9]. In the high-frequency and the collisionless limits, $k\lambda_{ei} \gg 6Z^{2/3}$, the moments (12) depend on the parameter ω/kv_T. They can be evaluated using $\nu_1 = kv_T h_1$, where $h_{l-1} = -i\omega/kv_T + x^2 l^2/(4l^2-1)h_l$ [cf. Equation (1)]. After substituting this expression into Eq. (12), we recover results derived by Bendib *et al.* [7]. In the collisionless quasistationary case $\omega \ll kv_T$, we obtain [1] $\{\sigma/e^2, \alpha/e, \chi/T\} = \{2.5, -1, 4\}nv_T/\sqrt{2\pi}kT$.

Because of its importance to laser produced plasmas, we apply Eq. (6) and our closure relations to the linear self-consistent evolution of a Gaussian temperature hot spot. The localized temperature perturbation, δT, relaxes along the dashed line in Fig. 2. Arrows define the direc-

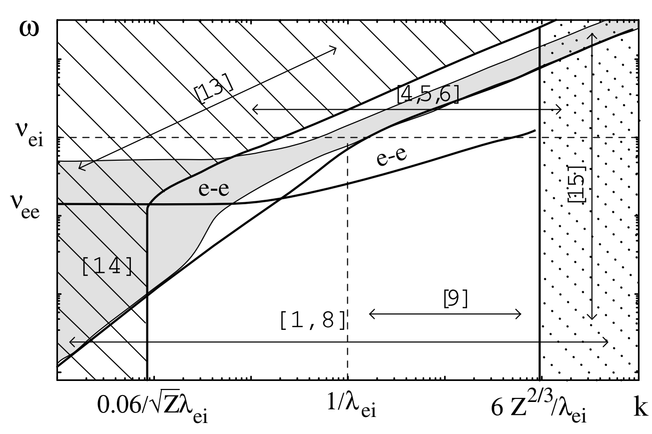

FIG. 4. The parametric plane for the electron permittivity. Numbers in brackets refer to the list of references.

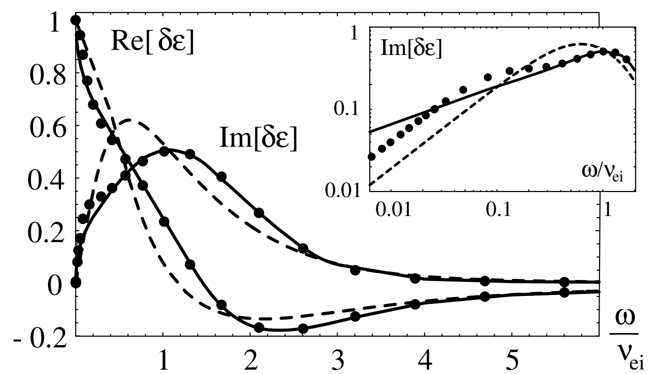

FIG. 5. The dependencies of the real and imaginary parts of $\delta\epsilon$ [Eq. (13)] (dots) on ω/kv_T for $k\lambda_{ei} = 1$ and $Z = 10$ in comparison with the theory [6] (solid lines) and Krook model (dashed lines).

tion of the evolution between points A and B as also shown in Fig. 3. Different rates of δT decay in Fig. 3 are due to the changing temperature gradient and the time-dependent transport coefficients. The closure relations (7), which are valid over the entire (ω, k) plane fully define the linear plasma response, including a dielectric permittivity ϵ. Using the definition $\epsilon = 1 + i4\pi j/\omega E \equiv 1 + \delta\epsilon/k^2\lambda_{De}^2$, where $\lambda_{\rm De}$ is the electron Debye length, we obtain

$$\delta\epsilon = \left[1 - \omega\left(\frac{e^2 n}{k^2 T\sigma} + \frac{2n(\sigma + e\alpha)^2}{\sigma^2(2k^2\kappa - 3i\omega n)}\right)\right]^{-1} \equiv 1 + i\omega J_N^N. \tag{13}$$

In combination with our solution to the FP equation, this is an explicit expression for the dielectric permittivity, which is valid over the entire region of $0 < k\lambda_{ei} < \infty$, $0 < \omega < \infty$, and for $Z \gg 1$. Until now, ϵ has been known only for certain domains of this parameter space. We have summarized variations of $\epsilon(\omega, k)$ in different regimes in Fig. 4 in a manner similar to Fig. 2 for transport coefficients. Figure 4 demonstrates the extension of classical collisional theory [13] and collisionless theory [15] to the entire domain of (ω, k). The gray region corresponds to strongly damped perturbations where $\text{Im}\epsilon > \text{Re}\epsilon$. Under the solid line in Fig. 4 (rising from the left bottom corner to the right top corner) the real part of the electron permittivity corresponds to Debye screening: $\text{Re}\epsilon = 1 + 1/(k^2\lambda_{\rm De}^2)$, and under the e-e curve, e-e collisions play an important role.

In the classical collisional limit, $k\lambda_{ei} < 0.06/\sqrt{Z}, 0.1\omega/\nu_{ei}$ (dashed domain in the Fig. 4), we can derive an analytical expression for $\delta\epsilon$ by substituting Eq. (9) into Eq. (13). In the quasistatic limit, $\omega < \nu_{ee}$ this gives the following expression ($x = 32k^2v_T^2/3\pi\nu_{ei}$):

$$\delta\epsilon = 2x(8x - 3i\omega)/(16x^2 - 6\omega^2 - 47i\omega x). \tag{14}$$

From the dispersion relation $\epsilon = 0$, one can derive the classical collisional entropy mode $\omega = 2ik^2\kappa/3n$ with the thermal conductivity coefficient defined through Eq. (10). In the high-frequency limit, $\omega \gg \nu_{ei}$, where the electrical conductivity provides the main contribution to the plasma response, we reproduce the well-known result, $\epsilon = 1 - (\omega_{pe}^2/\omega^2)(1 - i\nu_{ei}/\omega)$.

In the high-frequency case $\omega \gg \nu_{ee}, \nu_{ei}(k\lambda_{ei})^{4/7}/Z^{5/7}$, we recover longitudinal permittivity from Ref. [6], which smoothly transforms to the collisionless limit ($\nu_{ei} \to 0$), $\delta\epsilon = 1 + (\omega/\sqrt{2}kv_T)Z(\omega/\sqrt{2}kv_T)$, where $Z(x)$ is the standard plasma dispersion function. Figure 5 illustrates the frequency dependence of $\delta\epsilon$ in the weakly collisional regime, $k\lambda_{ei} = 1$, and compares results of the full FP equation solutions (dotted lines) with the plasma permittivity based on the Krook model calculations [11] (dashed lines). As described in Ref. [11], the particle density-conserving Krook model gives the most accurate results in this regime. Still, in the region $\omega \sim \nu_{ei}$, the deviation of the Krook model results from the exact solution can be as high as factor 2 or 3.

We have derived nonlocal closure relations (in both space and time) and dielectric permittivity for high-Z electron plasma. This theory is equivalent to the solution of the linear Fokker-Planck kinetic equation. It covers the entire (ω, k) space for arbitrary plasma collisionality. We have provided practical expressions for transport coefficients and dielectric permittivity which can be applied over the entire regime of parameters.

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