Determination of the a_0-a_2 Pion Scattering Length from $K^+ \to \pi^+\pi^0\pi^0$ Decay

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We present a new method for the determination of the π-π scattering length combination $a_0 - a_2$, based on the study of the $\pi^0\pi^0$ spectrum in $K^+ \to \pi^0\pi^0\pi^+$ in the vicinity of the $\pi^+\pi^-$ threshold. The method requires a minimum of theoretical input, and is potentially very accurate.

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Current algebra and partially conserved axial current lead to a prediction for the threshold behavior of π-π scattering [1,2]. The $I = 0$ and $I = 2$ S-wave scattering lengths were predicted to be $a_0 m_{\pi^+} = 0.159$, $a_2 m_{\pi^+} = -0.045$, a first approximation that can be improved upon in the framework of chiral perturbation theory [3]. Recent calculations [4,5], which combine ChPT with the dispersive approach by Roy [6,7], lead to

$$a_0 m_{\pi^+} = 0.220 \pm 0.005, \tag{1}$$

$$a_2 m_{\pi^+} = -0.0444 \pm 0.0010, \tag{2}$$

$$(a_0 - a_2) m_{\pi^+} = 0.265 \pm 0.004. \tag{3}$$

The current discussion of this prediction [8–10] could lead to minor modifications of Eqs. (1)–(3).

It was long recognized [11] that the angular distributions in $K^+ \to \pi^+\pi^- e^+\nu$ are sensitive to the $\pi\pi$ phase shifts, and can be used to obtain informations on the S-wave scattering lengths [12,13]. The first results by the Geneva-Saclay experiment [14], leading to $a_0 m_{\pi^+} = 0.26 \pm 0.05$, were recently improved by the E865 Collaboration at Brookhaven [15] that quotes a result: $a_0 m_{\pi^+} = 0.216 \pm 0.013(\mathrm{stat}) \pm 0.002(\mathrm{syst}) \pm 0.002(\mathrm{theor})$. Data on K_{e4}, with a large statistics, are currently being analyzed by the NA48 Collaboration at CERN.

The K_{e4} decay yields values of the phase shift difference $\delta_0^0 - \delta_1^1$ as a function of the $\pi\pi$ invariant mass $M_{\pi\pi}$ in the range $2m_{\pi^+} < M_{\pi\pi} < M_K - m_{\pi^+}$, but the best data lies in the range >310 MeV. The extraction of a value for a_0 requires an extrapolation to the threshold region and a substantial theoretical input, whence the interest in alternative methods which permit the determination of the scattering lengths through measurements that are directly sensitive to $\pi\pi$ scattering in the threshold region, $M_{\pi\pi} \sim 2m_{\pi^+}$. An example of this is the measurement of the $\pi^0\pi^0$ decay of the pionic atom $\pi^+\pi^-$, the object of the DIRAC experiment at CERN [16,17] that could yield a value for the a_0-a_2 combination.

I present here an alternative method for determining a_0-a_2, based on the $\pi^0\pi^0$ mass distribution in the $K^+ \to \pi^+\pi^0\pi^0$ decay in the vicinity of the $\pi^+\pi^-$ threshold. The large data sample available from the NA48 experiment at CERN, of the order of 10^8 events, could lead to a determination of a_0-a_2 with a precision comparable or higher than that foreseen in the DIRAC experiment. The method is based on the fact that the $K^+ \to \pi^+\pi^+\pi^-$ decay gives a contribution to the $K^+ \to \pi^+\pi^0\pi^0$ amplitude through the charge exchange reaction $\pi^+\pi^- \to \pi^0\pi^0$. This contribution is directly proportional to a_0-a_2, and displays a characteristic behavior when the $\pi^0\pi^0$ mass is in the vicinity of the $\pi^+\pi^-$threshold, where it goes from dispersive and real to absorptive and imaginary.

Let us write

$$\mathcal{M}(K^+ \to \pi^+\pi^0\pi^0) = \mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1, \tag{4}$$

where $\mathcal{M}_0$ is the "unperturbed amplitude," and $\mathcal{M}_1$ the contribution of the diagram in Fig. 1, with the renormalization condition

$$\mathcal{M}_1 = 0 \quad \text{for} \quad s_\pi = (q_1 + q_2)^2 = 4m_{\pi^+}^2. \tag{5}$$

The "unperturbed" amplitude $\mathcal{M}_0$, and the corresponding one $\mathcal{M}_+$ for $K^+ \to \pi^+\pi^+\pi^-$, can be parametrized as polynomials [18] in $s_i = (k - q_i)^2$. In both cases q_3 is chosen as the momentum of the "odd" pion, respectively π^+ and π^-. A simple parametrization, which gives a reasonable description of the experimental data, is given by

$$\mathcal{M}_0 = A_{\mathrm{av}}^0 (1 + g^0 (s_3 - s_0)/2m_{\pi^+}^2), \tag{6}$$

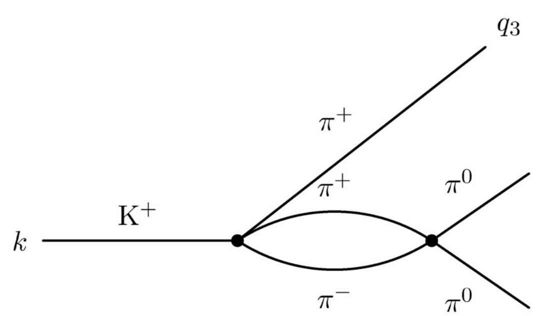

FIG. 1. The $\pi\pi$ rescattering diagram.

$$\mathcal{M}_+ = A^+_{\rm av}(1 + g^+(s_3 - s_0)/2m^2_{\pi^+}), \tag{7}$$

where $s_0 = (s_1 + s_2 + s_3)/3$. The g's coincide with the linear slope parameters defined in the Particle Data Group (PDG) review [18]. The $\Delta I = 1/2$ rule requires $A^0_{\rm av}$ and $A^+_{\rm av}$ to have the same sign [19], with $A^+_{\rm av} \sim 2A^0_{\rm av}$, in good agreement with the observed branching ratios. In the following we will assume $\mathcal{M}_0$ and $\mathcal{M}_+$ to be positive.

To evaluate the graph in Fig. 1 we can use a simplified effective Lagrangian which reproduces the $\pi\pi$ charge exchange reaction near the $\pi^+\pi^-$ threshold,

$$\mathcal{L}_{\rm chx} = \frac{16\pi(a_0 - a_2)m_{\pi^+}}{3}(\pi^+\pi^-\pi^0\pi^0). \tag{8}$$

The diagram in Fig. 1 then results in

$$\mathcal{M}_1 = -\frac{2(a_0 - a_2)m_{\pi^+}}{3}\mathcal{M}_{+,\rm thr}(J + K), \tag{9}$$

where $\mathcal{M}_{+,\rm thr}$ is the value of $\mathcal{M}_+$ at the $\pi^+\pi^-$threshold. Using Eq. (7),

$$\mathcal{M}_{+,\rm thr} = A^+_{\rm av}\left[1 + \frac{g^+(M^2_K - 9m^2_{\pi^+})}{12m^2_{\pi^+}}\right]. \tag{10}$$

We have divided the contribution of the graph into two parts, J and K. The J contribution flips from dispersive to absorptive at $s_{\pi\pi} = 4m^2_{\pi^+}$,

$$\begin{aligned} J = J_- &= \pi\tilde{v}: s_{\pi\pi} < 4m^2_{\pi^+}, \\ J = J_+ &= -i\pi v: s_{\pi\pi} > 4m^2_{\pi^+}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \tilde{v} &= ((4m^2_{\pi^+} - s_{\pi\pi})/s_{\pi\pi})^{1/2}, \\ v &= ((s_{\pi\pi} - 4m^2_{\pi^+})/s_{\pi\pi})^{1/2}. \end{aligned} \tag{12}$$

The K contribution is dispersive both above and below the threshold, and has no singularity at $s_{\pi\pi} = 4m^2_{\pi^+}$ so that it can be approximated by a polynomial in $s_{\pi\pi}$. It will be reabsorbed in the definition of the unperturbed amplitude $\mathcal{M}_0$, setting $K = 0$ in Eq. (9).

Since $\mathcal{M}^1$ changes from real to imaginary at the $\pi^+\pi^-$threshold, $|\mathcal{M}|^2$ will have a different expression below and above the threshold:

$$|\mathcal{M}|^2 = \begin{cases} (\mathcal{M}_0)^2 + (\mathcal{M}_1)^2 + 2\mathcal{M}_0\mathcal{M}_1 & : \text{below}, \\ (\mathcal{M}_0)^2 + (i\mathcal{M}_1)^2 & : \text{above}, \end{cases} \tag{13}$$

and the differential decay rate for $K^+ \to \pi^+\pi^0\pi^0$ with respect to the $\pi^0\pi^0$ invariant mass $M_{\pi\pi}$ will display a cusp. In Fig. 2 we show a plot of the differential decay rate (in arbitrary units) before and after the rescattering corrections, evaluated using $A^+_{\rm av} = 2A^0_{\rm av}$, the slope parameters g^0, g^+ as given in the PDG listings, and the value for a_0-a_2 from Eq. (3). The $(4m^2_{\pi^+} - s_{\pi\pi})^{1/2}$ behavior below the $\pi^+\pi^-$ threshold arises from the interference term in Eq. (13) and is a very characteristic feature. It is encouraging to see that the deviation from the uncorrected behavior is very prominent, so that it should be possible to measure it accurately.

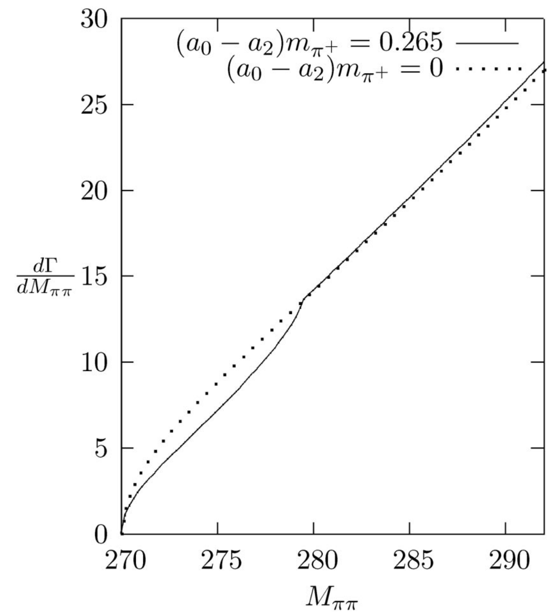

FIG. 2. The $\pi^0\pi^0$ invariant mass distribution with/without the rescattering correction, in arbitrary units.

In order to extract the value of a_0-a_2 from the $\pi^0\pi^0$ spectrum, let us consider a development of $|\mathcal{M}|^2$ in powers of $\delta = (4m^2_{\pi^+} - s_{\pi\pi})^{1/2}/2m_{\pi^+}$. Below the $\pi^+\pi^-$ threshold the coefficients of δ and of δ^2 are uniquely determined in terms of the rate for $K^+ \to \pi^+\pi^0\pi^0$ above this threshold, the $K^+ \to \pi^+\pi^+\pi^-$ differential rate, and the value of a_0-a_2. Since the maximum value of δ below threshold is $\sim$0.26, neglecting terms in δ^3 and higher is equivalent to a $\sim$2% theoretical error in the decay rate, corresponding to a $\sim$6% error on the value of a_0-a_2. This is the central result of this Letter, and it is worthwhile to discuss it in more detail.

Above the $\pi^+\pi^-$ threshold $\mathcal{M}_1$ is absorptive, so that its value is directly determined by the physical amplitudes for $K^+ \to \pi^+\pi^+\pi^-$ and $\pi^+\pi^- \to \pi^0\pi^0$. In Eqs. (9) and (11) we have neglected the $s_{\pi\pi}$ dependence of the charge exchange reaction and of the $K^+ \to \pi^+\pi^+\pi^-$ amplitude, which can contribute terms of $O(\delta^3)$ to $\mathcal{M}_1$. As noted before in the discussion of the K term, even powers of δ are absent from $\mathcal{M}_1$ because they can be absorbed in the definition of $\mathcal{M}_0$. The value of $\mathcal{M}_1$ below the threshold is the analytic continuation of the value above the threshold, so that it correctly includes the $O(\delta)$ terms, with possible errors which are $O(\delta^3)$.

Terms of $O[\delta^2 = (4m^2_{\pi^+} - s_{\pi\pi})/4m^2_{\pi^+}]$ in the value of $|\mathcal{M}|^2$, Eq. (13), derive from two sources: the first is in the $s_{\pi\pi}$dependence of $\mathcal{M}_0$—see, e.g., Eq. (6), keeping in mind that $s_3 = (k - q_3)^2 = (q_1 + q_2)^2 = s_{\pi\pi}$. Since $\mathcal{M}_0$ is regular at the threshold, the coefficient of this contribution is the same on either side of it. The second source of $O(\delta^2)$ terms is from the $(\mathcal{M}_1)^2$ terms in Eq. (13). In this case, since $\tilde{v}^2 = -v^2$, the coefficient of δ^2 changes sign across the threshold. This coefficient is predicted by Eqs. (9) and (11). We can thus proceed as follows:

(1) Measure $\mathcal{M}_{+,\mathrm{thr}}$ from the $K^+ \to \pi^+\pi^+\pi^-$ decay at the $\pi^+\pi^-$ threshold. In terms of the PDG inspired parametrization in Eq. (7), $\mathcal{M}_{+,\mathrm{thr}}$ is given by Eq. (10).

(2) Fit $|\mathcal{M}|^2 = (\mathcal{M}_0)^2 + (i\mathcal{M}_1)^2$, measured from $K^+ \to \pi^+\pi^0\pi^0$ with $M_{\pi\pi}$ above the $\pi^+\pi^-$ threshold, to a polynomial in δ^2, $|\mathcal{M}|^2 = F(\delta^2)$.

(3) $|\mathcal{M}|^2$ below the threshold will then be given by

$$|\mathcal{M}|^2 = F(\delta^2) + 2\mathcal{M}_1(F(\delta^2) + (\mathcal{M}_1)^2)^{1/2} + 2(\mathcal{M}_1)^2, \tag{14}$$

where $F(\delta^2)$ is the polynomial obtained in the second step.

(4) Using Eqs. (9) and (11), we can express $\mathcal{M}_1$ in terms of a_0-a_2, so that this quantity can be obtained by fitting the $\pi^0\pi^0$ spectrum below the $\pi^+\pi^-$ threshold to Eq. (15).

We have not so far discussed the contribution of the diagrams, similar to that in Fig. 1, which arise from the unperturbed amplitude $\mathcal{M}_0$ with $\pi^0\pi^0 \to \pi^0\pi^0$ or $\pi^+\pi^0 \to \pi^+\pi^0$ rescattering. These contributions are always absorptive, and generally smaller than $\mathcal{M}_1$, since they are proportional to $\mathcal{M}_0$, which is smaller than $\mathcal{M}_+$, and to smaller combinations of the pion scattering lengths, respectively $(a_0 + 2a_2)/3$ and a_2. For the $\pi^0\pi^0 \to \pi^0\pi^0$ rescattering one finds, e.g.,

$$\mathcal{M}_2 = i\frac{(a_0 + 2a_2)m_{\pi^+}}{3}\mathcal{M}_{0,\mathrm{thr}}\left(1 - \frac{4m_{\pi^0}^2}{s_{\pi\pi}}\right)^{1/2}, \tag{15}$$

where $\mathcal{M}_{0,\mathrm{thr}}$ is the unperturbed amplitude at the $\pi^0\pi^0$ threshold [20]. These amplitudes do not interfere with $\mathcal{M}_0$, but they interfere with $\mathcal{M}_1$ above the $\pi^+\pi^-$ threshold, giving rise to a small cusp which might be detected in high-statistics experiments. Their effects do not substantially alter our conclusions, but should be included in an analysis of the experimental data. The best strategy could be to accept for them the theoretical predictions from Eqs. (1)–(3), while extracting a value for $(a_0$-$a_2)$.

Although the method outlined here seems to require a minimum of theoretical elaboration, more theoretical work is needed. Given the possible precision of the method, it would be nice to obtain a more exact evaluation of the $O(\delta^3)$ corrections to $|\mathcal{M}|^2$. This will be possible with the methods of chiral perturbation theory. It is of course possible to account for these corrections by introducing an extra parameter in the fit to the experimental data. We might also wish to evaluate the electromagnetic corrections to our predictions.

We note that a similar cusp effect arises in the interference between $K_L \to \pi^0\pi^0\pi^0$ and $K_L \to \pi^+\pi^-\pi^0$ followed by $\pi^+\pi^- \to \pi^0\pi^0$. The effect is smaller than in Fig. 2, but could also lead to a determination of a_0-a_2. Similar effects should also appear in $\eta \to 3\pi^0$ decays, but this process is not competitive from an experimental point of view.

Threshold cusp phenomena have a long history [21,22]. They have been studied in $\pi^- P \to \Lambda K^0$ near the ΣK threshold [23,24] in an attempt to determine the relative Σ-Λ parity, and more recently [25] in $\gamma P \to \pi^0 P$ near the $N\pi^+$ threshold, where they can yield informations on the π-nucleon scattering lengths. In contrast to the phenomenon discussed here, the analysis of cusp phenomena in two-body processes is inherently more complicated.

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