Weak Measurements with Arbitrary Probe States

Lars M. Johansen*

Department of Technology, Buskerud University College, P.O. Box 251, N-3601 Kongsberg, Norway

(Received 5 February 2004; published 14 September 2004)

The exact conditions on valid probe states for weak measurements are derived. It is demonstrated that weak measurements can be performed with any probe state with vanishing probability current density. This condition is found both for weak measurements of noncommuting observables and for c-number observables. In addition, the interaction between object and probe must be sufficiently weak. Strange weak values can be observed also with mixed probe states, but not for c-number observables.

DOI: 10.1103/PhysRevLett.93.120402 PACS numbers: 03.65.Ta

In ‘‘orthodox’’ quantum mechanics, the result of a ‘‘measurement’’ is always one of the eigenvalues of the observable. In the last pages of his textbook on quantum mechanics [1], von Neumann provided a model of a measurement where the object under study was interacting with a measurement probe (or pointer). By assuming that the initial uncertainty of the probe was small, von Neumann demonstrated that the probe would display one of the eigenvalues of the object observable.

Aharonov, Albert, and Vaidman (AAV) considered the same experimental arrangement [2]. However, they made the opposite assumption, namely, that the initial uncertainty of the probe was large. They demonstrated that despite this, the probe would on average show the correct expectation value of an observable $\hat{c}$, although it could not distinguish separate eigenvalues. Their most interesting discovery, though, was that if a projective measurement of a second observable $\hat{d}$ was made on the *object* after the interaction, the average meter reading conditioned on the result of the measurement on the object would be the real part of the quantity

$$c_w(d) = \frac{\langle d \mid \hat{c} \mid \psi \rangle}{\langle d \mid \psi \rangle}. \tag{1}$$

The authors introduced the name ‘‘weak value’’ for this quantity. They observed that the values of c_w might lie outside the range of eigenvalues of the observable $\hat{c}$. It has been contested whether the experimental arrangement of AAV qualifies as a measurement, and whether it has any meaning to ascribe to c_w a significance as a ‘‘value’’ of the observable $\hat{c}$ [3–5].

Despite initial scepticism, weak values have found applications in a variety of systems. A classical optical analog [6] of the original experiment proposed by AAV has been realized experimentally [7]. The polarization state of a classical radiation field can be treated as analogous to a spin-$\frac{1}{2}$ system, and the weak value of polarization may exceed the eigenvalue range [8,9]. It has also been found that weak values have applications within classical optical communication [10,11]. These examples demonstrate that weak values have applications beyond quantum mechanics. In fact, it is not unfamiliar that the quantum formalism can be applied even to classical systems where two observables cannot be jointly measured with arbitrary accuracy. This is known, e.g., in signal processing, where the Wigner distribution for time and frequency has gained popularity [12].

It was recently found that weak values have a deeper significance when considered from the viewpoint of standard Bayesian estimation theory [13]. If the preselected state is considered as ‘‘prior information’’, an estimate can be made of the observable $\hat{c}$ on basis of the postselection measurement of the observable $\hat{d}$. It then turns out that the weak value is the most efficient estimator for the observable $\hat{c}$. One may, instead of performing a weak measurement between preselection and postselection, try to guess the value of $\hat{c}$. The best possible guess between preselection and postselection is nothing else than the weak value.

Furthermore, it can be shown [14] that the real part of the weak value can be expressed as a conditional moment of the Terletsky-Margenau-Hill distribution [15,16]. More generally, the weak value is a conditional moment of the standard ordered distribution [17], which is the complex conjugate of the Kirkwood distribution [18]. This is a consequence of the noncommutativity of the observables involved in weak measurements. One may also consider weak measurements of observables when they are treated as c numbers. In this case, it can be shown that the resulting weak value is a conditional moment of a classical joint distribution of the two observables involved. Weak values are simply the expectation value of one variable given the second variable.

There has been, and still is, considerable controversy over the interpretation of weak measurements and weak values. An inherit assumption in the discussions on weak measurements seems to be that the probe has to be in a pure state. For example, Aharonov and Vaidman say that [19] ‘‘of course, we can construct measurement with a large uncertainty which is not weak at all, for example, by preparing the measuring device in a mixed state instead of a Gaussian [...]’’. But is it really necessary that weak measurements must be performed with pure probe states? It is the purpose of this Letter to demonstrate that weak measurements can be performed with virtually any state of the probe. The only necessary restriction is that the current density of the probe must vanish. As long as this is satisfied, "strange" weak values that exceed the eigenvalue range of the observable can be observed using any state of the probe, pure or mixed. For example, a mixed, high temperature thermal probe will yield exactly the same weak values as a pure Gaussian probe state. Furthermore, we shall see that this conclusion is valid both for noncommuting and c-number observables. As a further consequence, we demonstrate that strange weak values that exceed the spectrum of the observable cannot be observed for c-number observables.

We first examine weak measurements of noncommuting observables. We consider an object and a probe described by the density operators $\hat{\rho}_s$ and $\hat{\rho}_a$, respectively. Prior to the measurement interaction, the combined object plus probe is assumed to be in a product state $\hat{\rho}_0 = \hat{\rho}_s \otimes \hat{\rho}_a$. We wish to perform a weak measurement of an arbitrary object observable $\hat{c}$. To this end, we shall assume that the interaction part of the Hamiltonian has the form

$$\hat{H}_\epsilon = \epsilon\delta(t)\hat{c} \otimes \hat{P}. \tag{2}$$

This interaction Hamiltonian is the one employed by von Neumann [1]. For a discussion on the reasons for using exactly this interaction term, see Ref. [20]. $\hat{P}$ is the momentum observable of the probe. We will consistently denote observables associated with the probe by capital letters. We assume that during the measurement interaction, the interaction part of the Hamiltonian dominates the time evolution. Nevertheless, we shall assume that the interaction between the object and probe is weak, i.e., ϵ is so small that we can perform a series expansion to first order in ϵ.

The time evolution is determined by (setting $\hbar = 1$)

$$\frac{\partial\rho}{\partial t} = -i[\hat{H}_\epsilon, \rho]. \tag{3}$$

Because of the interaction between the object and probe, the density operator evolves to $\hat{\rho}_\epsilon = \hat{U}_\epsilon\hat{\rho}_0\hat{U}_\epsilon^\dagger$. Since the Hamiltonian commutes with itself at all times, the evolution operator $\hat{U}_\epsilon$ can be written as

$$\hat{U}_\epsilon = e^{-i\int\hat{H}_\epsilon(t)dt} = e^{-i\epsilon\hat{c}\otimes\hat{P}}. \tag{4}$$

A projective measurement will be made of the probe position $\hat{Q}$ and an object observable $\hat{d}$. We therefore consider the joint probability distribution

$$\rho_\epsilon(Q, d) = \langle d \mid \otimes\langle Q \mid \hat{\rho}_\epsilon \mid Q\rangle\otimes \mid d\rangle. \tag{5}$$

We shall study $\rho_\epsilon(Q, d)$ for small ϵ. We therefore perform a Maclaurin expansion in ϵ. Using the fact that

$$\left(\frac{\partial^n\hat{U}_\epsilon}{\partial\epsilon^n}\right)_{\epsilon=0} = (-i)^n\hat{c}^n \otimes \hat{P}^n, \tag{6}$$

we obtain the expansion

$$\rho_\epsilon(Q, d) = \rho_0(Q, d) + i\epsilon(\langle d \mid \hat{\rho}_s\hat{c} \mid d\rangle\langle Q \mid \hat{\rho}_a\hat{P} \mid Q\rangle - \langle d \mid \hat{c}\hat{\rho}_s \mid d\rangle\langle Q \mid \hat{P}\hat{\rho}_a \mid Q\rangle) + R_\epsilon(Q, d) \tag{7}$$

where

$$\rho_0(Q, d) = \langle d \mid \hat{\rho}_s \mid d\rangle\langle Q \mid \hat{\rho}_a \mid Q\rangle \tag{8}$$

is the joint probability distribution for Q and d prior to the interaction, and where

$$R_\epsilon(Q, d) = \sum_{n=2}^{\infty}\frac{(i\epsilon)^n}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\langle d \mid \hat{c}^{n-k}\hat{\rho}_s\hat{c}^k \mid d\rangle \times\langle Q \mid \hat{P}^{n-k}\hat{\rho}_a\hat{P}^k \mid Q\rangle \tag{9}$$

is a "remainder term". The Lagrange form of the remainder term is

$$R_\epsilon(Q, d) = \frac{1}{2}\frac{d^2\rho_\xi(Q, d)}{d\xi^2}\epsilon^2 \tag{10}$$

for some ξ in the range $(0, \epsilon)$. It can be shown that an upper limit for the remainder term is given for $\xi = \epsilon$. The lowest order approximation to the remainder term is found for $\xi = 0$.

Assuming that the current density of the probe vanishes

$$\langle Q \mid \hat{P}\hat{\rho}_a \mid Q\rangle + \langle Q \mid \hat{\rho}_a\hat{P} \mid Q\rangle = 0, \tag{11}$$

it can be shown that

$$\langle Q \mid \hat{\rho}_a\hat{P} \mid Q\rangle = \frac{i}{2}\frac{\partial}{\partial Q}\langle Q \mid \hat{\rho}_a \mid Q\rangle. \tag{12}$$

We then find that

$$\rho_\epsilon(Q, d) = \rho_o(Q, d) - \epsilon\mathrm{Re}(c_w)\frac{\partial\rho_o(Q, d)}{\partial Q} + R_\epsilon(Q, d), \tag{13}$$

where

$$c_w(d) = \frac{\langle d \mid \hat{c}\hat{\rho}_s \mid d\rangle}{\langle d \mid \hat{\rho}_s \mid d\rangle} \tag{14}$$

can be recognized as the weak value of the observable $\hat{c}$ for an object preselected in a mixed state $\hat{\rho}_s$ and postselected in the eigenstate $\mid d\rangle$ [2,13].

By integrating Eq. (13), it is found that, to first order in ϵ, the probability density for the object observable d is unaffected by the measurement interaction,

$$\int dQ\rho_\epsilon(Q, d) \approx \int dQ\rho_0(Q, d) = \langle d \mid \hat{\rho}_s \mid d\rangle. \tag{15}$$

We may write the conditional probability density for the probe position as

$$\rho_\epsilon(Q \mid d) = \frac{\rho_\epsilon(Q, d)}{\int dQ \rho_\epsilon(Q, d)} \approx \hat{\mathcal{T}} \langle Q \mid \hat{\rho}_a \mid Q \rangle, \tag{16}$$

where

$$\hat{\mathcal{T}} = 1 - \epsilon \mathrm{Re}(c_w) \frac{\partial}{\partial Q} \tag{17}$$

is a first order translation operator. The probe position, given a value d of the object observable, has been translated by a distance $\epsilon \mathrm{Re}(c_w)$.

If the standard deviation of the probe position is σ, the basic condition for a weak measurement is that the translation of the probe should be small compared to the standard deviation of the probe, $| \epsilon \mathrm{Re}(c_w) | \ll \sigma$. For a comparison with the requirements for standard, projective measurements, see Ref. [20]. To be precise, also the second and higher order corrections to the expectation value of the probe position should be small compared to the first order change. This requires that

$$\left| \int dQ Q R_\epsilon(Q, d) \right| \ll |\epsilon \mathrm{Re}(c_w)| \langle d \mid \hat{\rho}_s \mid d \rangle. \tag{18}$$

We now turn to weak measurements of c-number observables. This is relevant, e.g., in classical mechanics and for classical radiation fields. We assume that the object and probe both can be described by a classical phase space distribution. Prior to the measurement interaction, we assume that the object plus probe is in a product state $F_0 = F_s(q, p) F_a(Q, P)$, where capital letters denote the probe. We consider a weak measurement of a general c-number object variable $c(q, p)$, and assume that the interaction Hamiltonian is

$$H_\epsilon = \epsilon \delta(t) c(q, p) P. \tag{19}$$

This is the c-number equivalent of the quantum interaction term (2). We assume that the interaction Hamiltonian dominates over any other terms in the Hamiltonian during the short time of interaction. The equation of motion is given by the classical Liouville's theorem

$$\frac{\partial F}{\partial t} = -\{F, H_\epsilon\}, \tag{20}$$

where

$$\{F, H_\epsilon\} = \sum_i \left(\frac{\partial F}{\partial q_i} \frac{\partial H_\epsilon}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H_\epsilon}{\partial q_i} \right). \tag{21}$$

For the Hamiltonian (19) we have

$$\frac{\partial H_\epsilon}{\partial p} = \epsilon \delta(t) \frac{\partial c(q, p)}{\partial p} P, \tag{22}$$

$$\frac{\partial H_\epsilon}{\partial q} = \epsilon \delta(t) \frac{\partial c(q, p)}{\partial q} P, \tag{23}$$

$$\frac{\partial H_\epsilon}{\partial P} = \epsilon \delta(t) c(q, p), \tag{24}$$

$$\frac{\partial H_\epsilon}{\partial Q} = 0. \tag{25}$$

The Poisson bracket therefore may be written as

$$\{H_\epsilon, F\} = \epsilon \delta(t) \left[P \left(\frac{\partial c}{\partial p} \frac{\partial}{\partial q} - \frac{\partial c}{\partial q} \frac{\partial}{\partial p} \right) + c \frac{\partial}{\partial Q} \right] F. \tag{26}$$

This has the form $\{F, H_\epsilon\} = \hat{\mathcal{H}}_\epsilon F$. Liouville's theorem (20) then can be written in a form similar to the Schrödinger equation,

$$\frac{\partial F}{\partial t} = -\hat{\mathcal{H}}_\epsilon F. \tag{27}$$

The state after the measurement interaction can then be expressed as $F_\epsilon = \hat{\mathcal{U}}_\epsilon F_0$, where the classical propagator is

$$\hat{\mathcal{U}}_\epsilon = e^{-\int \hat{\mathcal{H}}_\epsilon dt}. \tag{28}$$

In the particular problem considered here, we write $\hat{\mathcal{H}}_\epsilon = \epsilon \delta(t) \hat{\mathcal{K}}$, where

$$\hat{\mathcal{K}} = P \left(\frac{\partial c}{\partial p} \frac{\partial}{\partial q} - \frac{\partial c}{\partial q} \frac{\partial}{\partial p} \right) + c \frac{\partial}{\partial Q}. \tag{29}$$

The propagator (28) then simplifies to

$$\hat{\mathcal{U}}_\epsilon = e^{-\epsilon \hat{\mathcal{K}}}. \tag{30}$$

We consider this propagator to first order in ϵ, $\hat{\mathcal{U}}_\epsilon \approx 1 - \epsilon \hat{\mathcal{K}}$. After the measurement interaction, the joint probability density for Q and q is

$$\rho_\epsilon(Q, q) = \int dp \int dP F_\epsilon(q, p, Q, P). \tag{31}$$

We then find that

$$\begin{aligned} \rho_\epsilon(Q, q) \approx \rho_0(Q, q) &+ \epsilon \int dp \left[\frac{\partial c(q, p)}{\partial q} \frac{\partial F_s(q, p)}{\partial p} \right. \\ &\left. - \frac{\partial c(q, p)}{\partial p} \frac{\partial F_s(q, p)}{\partial q} \right] \int dP P F_a(Q, P) \\ &- \epsilon \int dp c(q, p) F_s(q, p) \frac{\partial f_a(Q)}{\partial Q}, \end{aligned} \tag{32}$$

where

$$\rho_0(Q, q) = f_s(q) f_a(Q), \tag{33}$$

$$f_s(q) = \int dp F_s(q, p), \tag{34}$$

$$f_a(Q) = \int dp F_a(Q, P). \tag{35}$$

We again assume that the current density of the probe vanishes,

$$\int dPPF_a(Q, P) = 0. \tag{36}$$

It then follows that to the first order in ϵ,

$$\rho_\epsilon(Q, q) = \hat{\mathcal{T}}_c \rho_0(Q, q), \tag{37}$$

where

$$\hat{\mathcal{T}}_c = 1 - \epsilon c_w \frac{\partial}{\partial Q} \tag{38}$$

is once more a translation operator, and where

$$c_w(q) = \frac{\int dpc(q, p)F_s(q, p),}{\int dpF_s(q, p)} \tag{39}$$

is the weak value of the c-number observable $c(q, p)$. We see that this is the conditional expectation value of $c(q, p)$. In other words, c_w is simply the expectation value of $c(q, p)$ "given" q. It follows trivially from this equation that the value of c_w must lie within the range of $c(q, p)$. In other words, strange weak values cannot be found for c-number observables.

By integrating Eq. (37) over Q, it follows that

$$f_s(q) = \int dQ\rho_\epsilon(Q, q). \tag{40}$$

This shows that to the first order in ϵ, the probability density of q is unaffected by the measurement interaction. The conditional probability density for the probe position then reads

$$\rho_\epsilon(Q \mid q) = \frac{\rho_\epsilon(Q, q)}{\int dQ\rho_\epsilon(Q, q)} = \hat{\mathcal{T}}_c f_a(Q). \tag{41}$$

This shows that the probe position Q has been translated by a distance ϵc_w. Also in this case, the measurement can be considered to be weak if $| \epsilon c_w | \ll \sigma$, where σ is the initial position uncertainty of the probe.

In conclusion, it was found that weak measurements can be performed with a much wider class of probe states than thought previously. Any state of the probe can be used provided that the current density of the probe vanishes. This conclusion is valid regardless of whether the observables are noncommuting or whether they are c numbers.

*Electronic address: lars.m.johansen@hibu.no

[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University, Princeton, 1955).
[2] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[3] A. J. Leggett, Phys. Rev. Lett. **62**, 2325 (1989).
[4] A. Peres, Phys. Rev. Lett. **62**, 2326 (1989).
[5] Y. Aharonov and L. Vaidman, Phys. Rev. Lett. **62**, 2327 (1989).
[6] I. M. Duck, P. M. Stevenson, and E. C. G. Sudarshan, Phys. Rev. D **40**, 2112 (1989).
[7] N. W. M. Ritchie, J. G. Story, and R. G. Hulet, Phys. Rev. Lett. **66**, 1107 (1991).
[8] J. M. Knight and L. Vaidman, Phys. Lett. A **143**, 357 (1990).
[9] A. D. Parks, D. W. Cullin, and D. C. Stoudt, Proc. R. Soc. London A **454**, 2997 (1998).
[10] N. Brunner, A. Acin, D. Collins, N. Gisin, and V. Scarani, Phys. Rev. Lett. **91**, 180402 (2003).
[11] D. R. Solli, C. F. McCormick, R. Y. Chiao, S. Popescu, and J. M. Hickmann, Phys. Rev. Lett. **92**, 043601 (2004).
[12] L. Cohen, *Time-Frequency Analysis* (Prentice-Hall, New Jersey, 1995).
[13] L. M. Johansen, Phys. Lett. A **322**, 298 (2004).
[14] L. M. Johansen, Phys. Lett. A **329**, 184 (2004).
[15] Y. P. Terletsky, Zh. Eksp. Teor. Fiz. **7**, 1290 (1937).
[16] H. Margenau and R. N. Hill, Prog. Theor. Phys. **26**, 722 (1961).
[17] C. L. Mehta, J. Math. Phys. (N.Y.) **5**, 677 (1964).
[18] J. G. Kirkwood, Phys. Rev. **44**, 31 (1933).
[19] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics*, edited by J. G. Muga, R. Sala Mayato, and I. L. Egusquiza, Lecture Notes in Physics Vol. M72 (Springer-Verlag, Berlin, 2002), p. 369.
[20] F. Haake and D. F. Walls, Phys. Rev. A **36**, 730 (1987).