Desynchronization Waves and Localized Instabilities in Oscillator Arrays

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We consider a ring of identical or near-identical coupled periodic oscillators in which the connections have randomly heterogeneous strength. We use the master stability function method to determine the possible patterns at the desynchronization transition that occurs as the coupling strengths are increased. We demonstrate Anderson localization of the modes of instability and show that such localized instability generates waves of desynchronization that spread to the whole array. Similar results should apply to other networks with regular topology and heterogeneous connection strengths.

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In this Letter, we discuss the synchronization of a large number of near-identical oscillators that are locally coupled with connections of random strength. Synchronization in networks of coupled oscillators has recently received considerable interest [1] and has relevance in fields such as biology [2], chemistry [3], lasers [4,5], and communications [6]. Usually, the networks studied have been assumed to have connections of equal strength. In practice, the connections between different oscillators may have different strengths, and in some cases this strength could have a large spread (e.g., in biological systems). A model and analysis method has been proposed by Pecora and Carroll [7] to systematically determine the stability of the synchronized state in a network of identical coupled oscillators. This method, the *master stability function*, has been used to study the synchronization properties of different networks [8,9]. Deng *et al.* [10] have obtained, using the master stability function technique, conditions for the distribution of the connection strengths that yield average stability of the synchronized state. Galias and Ogorzalek [11] have studied the effect of adding small perturbations to the coupling strengths in relatively small arrays of coupled chaotic oscillators. Denker *et al.* [12] have studied the effect of small coupling strength heterogeneity in networks of pulse-coupled oscillators. Our approach in this Letter will be different: we consider the coupling strengths to have a relatively large spread and will discuss phenomena that can be expected when a large number of periodic oscillators are coupled in such a network. In particular, we will see that, as the coupling strength is increased, the oscillators desynchronize in a localized region. The localization results because the connection matrix has random components and the eigenvectors of this matrix are Anderson localized [13,14]. The effect of the localized instability spreads as a wave throughout the array, eventually resulting in an ordered state. Remarkably, in the case where the oscillators are not identical, the final state of the locally unstable system was found to be, for the system we considered, more ordered than in the case where the system is stable.

We consider a model system of N identical dynamical units, each one of which, when isolated, satisfies $\dot{X}_i = F(X_i)$, where $i = 1, 2, \ldots, N$, and X_i is the d-dimensional state vector for unit i. (The case of nearly identical units is considered at the end of this Letter. See also [15].) The oscillators, when coupled, are taken to satisfy (e.g., [7])

$$\dot{X}_i = F(X_i) - g \sum_{j=1}^{N} G_{ij} H(X_j), \tag{1}$$

where the coupling function H is independent of i and j, and the matrix G is a symmetric Laplacian matrix ($\sum_j G_{ij} = 0$) describing the network connections. The constant g determines the global strength of the coupling.

There is an exactly synchronized solution of Eq. (1), $X_1 = X_2 = \cdots = X_N = s(t)$, whose time evolution is the same as the uncoupled dynamics of a single unit, $\dot{s} = F(s)$. In this Letter we will be concerned with the case where the synchronized state is *periodic*, $s(t + T) = s(t)$. The stability of the synchronized state can be determined from the variational equations obtained by considering an infinitesimal perturbation ϵ_i from the synchronous state, $X_i(t) = s(t) + \epsilon_i(t)$,

$$\dot{\epsilon}_i = DF(s)\epsilon_i - g \sum_{j=1}^{N} G_{ij} DH(s)\epsilon_j. \tag{2}$$

Let $\epsilon = [\epsilon_1, \epsilon_2, \ldots, \epsilon_N]$, and define the $d \times N$ matrix $\eta = [\eta_1, \eta_2, \ldots, \eta_N]$ by $\epsilon = \eta L^T$, where L is the orthogonal matrix whose columns are the corresponding real orthonormal eigenvectors of G; $GL = L\Lambda$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$ where λ_k is the eigenvalue of G for eigenvector k. Then Eq. (2) is equivalent to

$$\dot{\eta}_k = [DF(s) - g\lambda_k DH(s)]\eta_k. \quad (3)$$

The quantity η_k is the weight of the kth eigenvector of G in the perturbation ϵ. The linear stability of each "spatial" mode k is determined by the stability of the zero solution of (3). By introducing a scalar variable $\alpha = g\lambda_k$, the set of equations given by (3) can be encapsulated in the single equation,

$$\dot{\eta} = [DF(s) - \alpha DH(s)]\eta. \quad (4)$$

The master stability function $\Psi(\alpha)$ [7] associated with Eq. (4) is its largest Lyapunov exponent [or equivalently for our case of periodic $s(t)$, the largest real part of its Floquet exponents]. This function depends only on the coupling function H and the chaotic dynamics of an individual uncoupled element, but not on the network connectivity. The network connectivity determines the eigenvalues λ_k (independent of details of the dynamics of the chaotic units). The stability of the synchronized state of the network is determined by $\Psi_* = \sup_k \Psi(g\lambda_k)$, where $\Psi_* > 0$ indicates instability.

As an illustrative example, we consider periodic Rössler oscillators, obeying the equations

$$\dot{x} = -(y + z), \qquad \dot{y} = x + 0.2y,$$
$$\dot{z} = 0.2 + z(x - 2.5). \quad (5)$$

In terms of our previous notation, $d = 3$, and $X = [x, y, z]^T$. The master stability function for this system is shown in Fig. 1. As seen in this figure, $\Psi(\alpha)$ approaches zero from negative values as $\alpha \to 0^+$. This is a general feature for systems where the individual, uncoupled units are stable limit cycle oscillators. We also see that $\Psi(\alpha)$ crosses from negative (stable) values to positive (unstable) values at a critical α value ($\alpha \approx 4.15$). The existence of such a transition is a robust feature that depends on the type of coupling and oscillator.

We now consider a network of N of these oscillators nearest-neighbor coupled in a ring, such that the strength of each individual link is random. The coupling strengths are obtained from an independent and identically distributed random sequence $\{a_i\}_{i=1}^N$. The matrix G is then

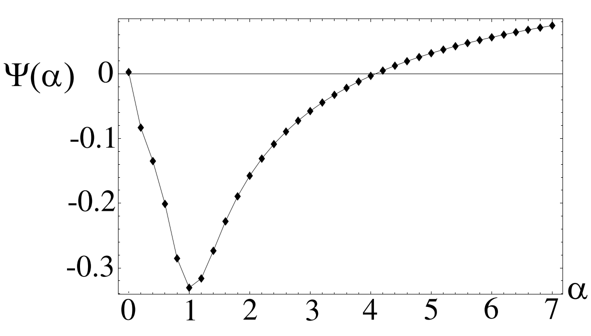

FIG. 1. Master stability function $\Psi(\alpha)$ versus α for Eqs. (5).

$$G = \begin{pmatrix} b_1 & -a_1 & 0 & 0 & \cdots & 0 & -a_N \\ -a_1 & b_2 & -a_2 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & b_3 & -a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_N & 0 & 0 & 0 & 0 & -a_{N-1} & b_N \end{pmatrix}, \quad (6)$$

where $b_i = (a_{i-1} + a_i)$ for $i = 1, \ldots, N$ (we take $a_0 \equiv a_N$).

The eigenvectors of the matrix G determine the possible desynchronization patterns. It is known that the eigenvectors of certain types of random matrices are exponentially localized (e.g., Anderson localization [13,14]). In our case, the eigenvector $\{u_i\}_{i=1}^N$ with eigenvalue λ satisfies

$$t_{i+1} = a_{i+1}^{-1}(\lambda + a_i + a_{i+1} - a_i t_i^{-1}), \quad (7)$$

where $t_i \equiv u_i/u_{i-1}$. Viewing Eq. (7) as a random dynamical system for t_i, we find numerically that, in our case,

$$\gamma = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \log(|t_i|) \quad (8)$$

exists and is independent of the initial condition and noise realization. Eigenvectors of (6) tend to have a localized amplitude peak at some location i_0 and decay as $|u_i| \propto e^{\gamma|i-i_0|}$ away from the peak; γ^{-1} is thus the localization length. (See [14].)

We choose the a_i's to be uniformly distributed in (0.1,1) (note that any multiple of this would lead to the same eigenvectors). (Since $a_i \geq 0.1$, we avoid the possibility $a_i \ll 1$ that would effectively disconnect the network.) The effects we will describe for this network should be regarded as an example of what could be expected in more general networks with random coupling. In Fig. 2(a) we show the eigenvector with the largest eigenvalue for a realization of the matrix G using $N = 500$. Figure 2(b) shows the localization length γ^{-1} as a function of λ calculated using Eq. (8). The eigenvectors are seen to be

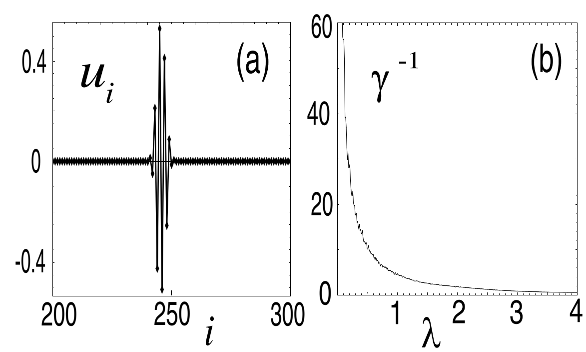

FIG. 2. (a) Eigenvector u_i for the largest eigenvalue $\lambda = 3.61$ for a particular realization of the matrix G in (6) with $N = 500$. (b) Localization length γ^{-1} calculated using Eq. (8).

sharply localized for the largest eigenvalues, and become less localized as the eigenvalues decrease.

As the coupling strength g is increased, the eigenvectors with the largest eigenvalue become unstable. These eigenvectors have the smallest localization length [see Fig. 2(b)]. We will now describe what occurs in this situation. We fixed the same realization of the matrix G used in producing Fig. 2(a). The four largest eigenvalues are 3.61, 3.41, 3.38, and 3.30. For $g = 1.24$ the eigenvector with the largest eigenvalue is unstable, and the next two eigenvectors are barely unstable ($\alpha = 4.47$, 4.23, and 4.19 in Fig. 1). We start with initial conditions near the synchronized state and then let the system evolve according to Eq. (1). In Fig. 3 we show snapshots of x_i as a function of the site index i for six successively increasing times.

Starting from a nearly synchronized state [Fig. 3(a)], the oscillators desynchronize at the location [see Fig. 2(a)] of the localized mode [Fig. 3(b)]. The desynchronization spreads as a wave to farther regions of the array [Figs. 3(c)–3(e)]. At the end, the domain of the wave covers the entire array [Fig. 3(f)]. This process is dominated by the most unstable mode. The other two less unstable modes can be seen as tiny defects at $i \approx 327$, 402 in the otherwise smooth wave. (The effect of these less unstable modes is most evident in Fig. 3(c). They also have a discernible, although small, effect in the final state [arrows in Fig. 3(f)].)

The final state and the process leading to it can be understood in terms of the phase of the oscillators. Define the phase $\phi(i,t) \equiv 2\pi\{n(i,t) + [t - t_-(i,t)] \times [t_+(i,t) - t_-(i,t)]^{-1}\}$, where $t_-(i,t) = \max\{s : x_i(s) = 0, \dot{x}_i > 0, s \leq t\}$, $t_+(i,t) = \min\{s : x_i(s) = 0, \dot{x}_i > 0, s > t\}$, and $n(i,t)$ is an integer chosen so that ϕ is a continuous function of t and that $\phi(i+1,t)$ is close to $\phi(i,t)$ for all i. Figure 4 shows two snapshots of the x coordinate and the phase as defined above as a function of i (the i origin was displaced so that what happens opposite the location of the unstable mode can be observed clearly, and for each time a constant was added to ϕ so that $\max_i \phi = 0$). As can be observed in the Figs. 4(a) and 4(c), a region with a constant phase gradient expands on both sides of the unstable mode. In the final state [Figs. 4(b) and 4(d)] the phase has a minimum at the location of the unstable mode and increases linearly on both sides reaching a maximum at the opposite end of the ring. This phase profile increases uniformly with time.

The cause of this phenomenon is that, as the oscillators in the region of the unstable mode desynchronize, they go to limit cycles that have a slightly lower frequency than that of the original orbit. Oscillating at a slower pace than the others, they drag the adjacent oscillators, and these drag theirs in turn, continuing until an equilibrium is reached. An equation describing approximately the evolution of the phase of the oscillator at location ξ and time t, $\phi(\xi, t)$, in a chain of diffusively coupled oscillators is given in the continuous limit by [16]

$$\frac{\partial \phi}{\partial t} = a\frac{\partial^2 \phi}{\partial \xi^2} + b\left(\frac{\partial \phi}{\partial \xi}\right)^2 + w(\xi), \tag{9}$$

where $w(\xi)$ is the frequency of the oscillator at location ξ, and a and b are constants. If this frequency is sufficiently smaller (larger) in a localized region and b is negative (positive), the equation predicts the development of waves that emanate from that region. The phase profile resulting from such forcing in a small region centered at the origin ($|\xi| < l$) can be approximated for large ξ and t as [16]

$$\phi(\xi, t) = w_0 t - \max[0, k(vt - |\xi|)], \tag{10}$$

where $w_0 = w(\xi)$ for $|\xi| > l$, and k and v depend on a and

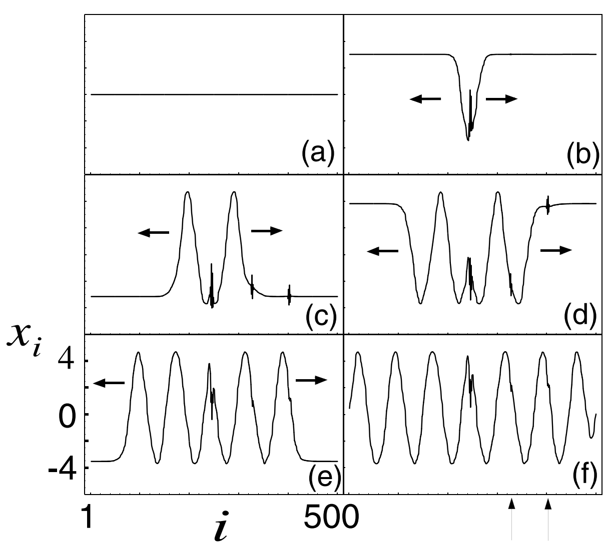

FIG. 3. Plots of the x coordinate of oscillator i versus the site index i, at times (a) 0, (b) 1400, (c) 2800, (d) 4200, (e) 5600, and (f) 10 000. All the plots have the same scale as (e).

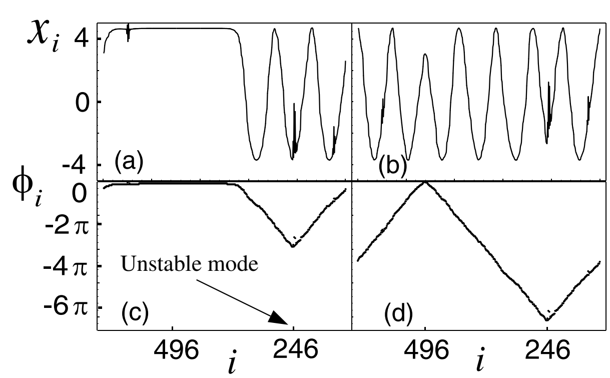

FIG. 4. Plots (a) and (b) show the x coordinate of oscillator i versus the site index i for times 3750 and 9660. Plots (c) and (d) show the phase of oscillator i at the same times as for (a) and (b), respectively. Compare with Eq. (10).

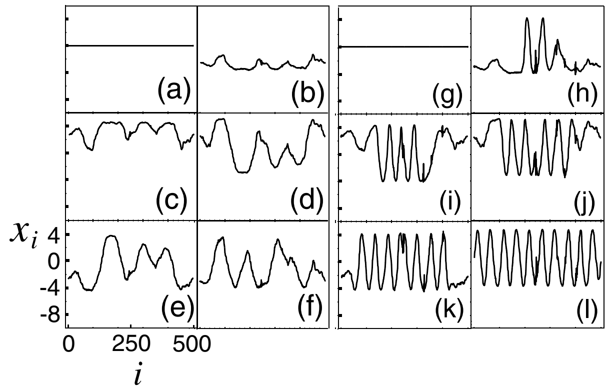

FIG. 5. Each plot shows the x coordinate of oscillator i as a function of the site index i. The time is 0, 1400, 2800, 4200, 5600, and 9970 for plots (a) to (f) and similarly for plots (g) to (l). A parameter mismatch was introduced in the oscillators. (a)–(f): All the modes are stable. (g)–(l): The pattern is organized by an unstable mode as in Fig. 3(f). All the plots have the same scale as (e).

b and $w(\xi)$. For appropriate k and v, Eq. (10) agrees well with Figs. 4(c) and 4(d).

In the example presented above, the pattern created by the unstable mode can be regarded as a more disordered synchronization than that of the original identical synchronization. However, in realistic situations, an unstable mode can actually make synchronization more orderly. In real systems, small differences in the parameters or small noise are expected. Under these circumstances, the different oscillators will be subject to small perturbations. The modes with eigenvalues close to zero have a master stability function close to zero (see Fig. 1) and also are nearly unlocalized [see Fig. 2(b)]. Thus, the phase of each oscillator will be subject to perturbations whose projection onto the nearly unlocalized modes are only very weakly damped. The identical synchronization of the array is thus spoiled by mismatch or noise. As an illustration, we randomly perturb the parameters of the different oscillators, so that they lie within $\pm 3\%$ of the original parameters. We then solved Eq. (1) with $g = 1.1$ and $g = 1.24$. For $g = 1.1$, all the modes are stable; in the case $g = 1.24$, three modes are stable as discussed above. In Figs. 5(a)–5(f) we show snapshots of the case $g = 1.1$, and in Figs. 5(g)–5(l) we show the corresponding snapshots for the case $g = 1.24$. When all of the modes are stable, the system exhibits a state in which there is erratic slow variation of the x_i with i. When there is an unstable mode, however, a more organized state is reached. If one picks two different oscillators j and k, they will satisfy asymptotically $X_j(t - \tau) = X_k(t)$, where τ is a simple function of j and k [see Fig. 4(d)]. Thus the oscillators are pairwise lag synchronized [17]. In realistic large arrays of periodic oscillators, it might be convenient to have one unstable mode. Our results suggest that this mode could, despite its localized nature, induce global organization of the system (Fig. 5).

In conclusion, we find that large arrays of periodic oscillators locally coupled by connections of randomly heterogeneous strength can experience a desynchronization transition characterized by the appearance of unstable Anderson localized modes. Furthermore, we find that, past the transition, the localized mode might play the key role in organizing the final global pattern of the system oscillations.

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