Transverse Structure of Nucleon Parton Distributions from Lattice QCD

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This work presents the first calculation in lattice QCD of three moments of spin-averaged and spin-polarized generalized parton distributions in the proton. It is shown that the slope of the associated generalized form factors decreases significantly as the moment increases, indicating that the transverse size of the light-cone quark distribution decreases as the momentum fraction of the struck parton increases.

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I. Introduction.—The technique of choice to precisely study complex many-particle systems in many areas of physics is to use a known, weakly interacting external probe to measure a response function or correlation function $\langle \mathcal{O}(x,t)\mathcal{O}(0,0)\rangle$, such as the dynamic structure factor in condensed matter systems. Because asymptotic freedom renders QCD perturbative for high energy lepton scattering, the correlation function that naturally reveals the quark and gluon structure of the nucleon is the nucleon matrix element of the light-cone operator $\mathcal{O}_q(x)$ that annihilates and creates a quark at positions separated by a distance λ along the light cone:

$$\mathcal{O}_q(x) = \int \frac{d\lambda}{4\pi} e^{i\lambda x}\bar{q}\left(-\frac{\lambda}{2}n\right)\not{n}\mathcal{P}e^{-ig\int_{-\lambda/2}^{\lambda/2} d\alpha n\cdot A(\alpha n)} q\left(\frac{\lambda}{2}n\right), \tag{1}$$

where n is a unit vector along the light cone and A is the gauge field. Introductory[1] and comprehensive[2] reviews provide the background for this Letter, and the essential ideas are briefly summarized here. The familiar quark distribution $q(x)$ specifying the probability of finding a quark carrying a fraction x of the nucleon's momentum in the light-cone frame is measured by the diagonal nucleon matrix element, $\langle P|\mathcal{O}(x)|P\rangle = q(x)$. Expanding $\mathcal{O}(x)$ in local operators via the operator product expansion generates the tower of twist-two operators,

$$\mathcal{O}_q^{\{\mu_1\mu_2\cdots\mu_n\}} = \bar{q}\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}\cdots i\overleftrightarrow{D}^{\mu_n\}}q, \tag{2}$$

and the diagonal matrix element $\langle P|\mathcal{O}_q^{\{\mu_1\mu_2\cdots\mu_n\}}|P\rangle$ specifies the $(n-1)^{\mathrm{th}}$ moment of the quark distribution $\int dx x^{n-1}q(x)$.

The generalized parton distributions $H(x,\xi,t)$ and $E(x,\xi,t)$ [3–5] are measured by off-diagonal matrix elements of the light-cone operator

$$\langle P'|\mathcal{O}_q(x)|P\rangle = \langle\langle \not{n}\rangle\rangle H(x,\xi,t) + \frac{i\Delta_\nu}{2m}\langle\langle \sigma^{\mu\nu}n_\mu\rangle\rangle E(x,\xi,t), \tag{3}$$

where $\Delta^\mu = P'^\mu - P^\mu$, $t=\Delta^2$, $\xi = -n\cdot\Delta/2$, $\langle\langle\Gamma\rangle\rangle = \bar{u}(P')\Gamma u(P)$, and $u(P)$ is a Dirac spinor. Off-diagonal matrix elements of the twist-two operators $\langle P'|\mathcal{O}_q^{\{\mu_1\mu_2\cdots\mu_n\}}|P\rangle$ yield moments of the generalized parton distributions, which for $\xi=0$, are

$$\begin{aligned}\int dx x^{n-1}H(x,0,t) &= A_{n0}(t)\\ \int dx x^{n-1}E(x,0,t) &= B_{n0}(t),\end{aligned} \tag{4}$$

where $A_{ni}(t)$ and $B_{ni}(t)$ are referred to as generalized form factors (GFFs).

Analogous expressions in which the light-cone operator $\mathcal{O}_q(x)$ and twist-two operators contain an additional γ_5 measure the longitudinal spin density, $\Delta q(x)$ and spin-dependent generalized parton distributions $\tilde{H}(x,\xi,t)$ and $\tilde{E}(x,\xi,t)$ with moments $\tilde{A}_{ni}(t)$ and $\tilde{B}_{ni}(t)$. In this work, we present calculations of the generalized form factors $A_{(n=1,2,3),0}(t)$ and $\tilde{A}_{(n=1,2,3),0}(t)$ in full QCD and discuss their physical significance.

II. Transverse structure of parton distributions.—In general, $H(x,\xi,t)$ is complicated to interpret physically because it combines features of both parton distributions and form factors, and depends on three kinematical variables: the momentum fraction, x, the longitudinal component of the momentum transfer, ξ, and the total momentum transfer squared, t. In the particular case in which $\xi=0$, however, Burkardt [6] has shown that $H(x,0,t)$, as well as its spin-dependent counterpart $\tilde{H}(x,0,t)$, has a simple and revealing physical interpretation.

It is useful to consider a mixed representation in which transverse coordinates are specified in coordinate space, the longitudinal coordinate is specified in momentum space, and one uses light-front coordinates for the longitudinal and time directions: $x^{\pm} = (x^0 \pm x^3)/\sqrt{2}$, $p^{\pm} = (p^0 \pm p^3)/\sqrt{2}$. Using these variables, letting x denote the momentum fraction and $\vec{b}_\perp$ denote the transverse displacement (or impact parameter) of the light-cone operator relative to the proton state, one may define an impact parameter-dependent parton distribution in light-cone gauge

$$q(x, \vec{b}_\perp) \equiv \langle P^+, \vec{R}_\perp = 0, \lambda | \mathcal{O}_q(x, \vec{b}_\perp) | P^+, \vec{R}_\perp = 0, \lambda \rangle, \tag{5}$$

where

$$\mathcal{O}_q(x, \vec{b}_\perp) = \int \frac{dx^-}{4\pi} e^{ixp^+x^-} \bar{q}\Big(-\frac{x^-}{2}, \vec{b}_\perp\Big) \gamma^+ q\Big(\frac{x^-}{2}, \vec{b}_\perp\Big). \tag{6}$$

Burkardt shows that the generalized parton distribution $H(x, 0, t)$ is the Fourier transform of the impact parameter-dependent parton distribution, so that

$$\begin{aligned} H(x, 0, -\Delta_\perp^2) &= \int d^2 b_\perp q(x, \vec{b}_\perp) e^{i\vec{b}_\perp \cdot \vec{\Delta}_\perp}, \\ A_{n0}(-\Delta_\perp^2) &= \int d^2 b_\perp \int dx x^{n-1} q(x, \vec{b}_\perp) e^{i\vec{b}_\perp \cdot \vec{\Delta}_\perp}, \end{aligned} \tag{7}$$

where $A_{n0}(-\Delta_\perp^2)$ the second form follows from Eq. (4). Although one normally only expects a form factor to reduce to a Fourier transform of a density in the nonrelativistic limit, Ref. [6] shows that special features of the light-cone frame also produce this simple result in relativistic field theory. Thus, $H(x, 0, t)$ specifies how the transverse distribution of quarks varies with the longitudinal momentum fraction x.

Physically, we expect the transverse size of the nucleon to depend significantly on x. Averaging $q(x, \vec{b}_\perp)$ over all x produces $A_{1,0}(t)$ and thus corresponds to the form factor. Hence, the average size is characterized by the transverse rms radius $\langle r_\perp^2 \rangle^{1/2} = \langle x_1^2 + x_2^2 \rangle^{1/2} = \sqrt{\frac{2}{3}} \langle r^2 \rangle^{1/2}$. From the experimental electromagnetic form factor, the transverse rms charge radius of the proton is 0.72 fm. As $x \to 1$, a single active parton carries all the momentum and the spectator partons give a negligible contribution. In this case, the active parton represents the (transverse) center of momentum, and the distribution in impact parameter reduces to a delta function $\delta^2(\vec{b}_\perp)$ with zero spatial extent. Indeed, explicit light-cone wave functions [7,8] bear out this expectation, with the result [9]

$$\begin{aligned} q(x, \vec{b}_\perp) = (4\pi)^{n-1} \sum_{n,c} \sum_a \int \Big[\prod_{j=1}^n dx_j d^2 r_{\perp j} \Big] \\ \times \delta\Big(1 - \sum_{j=1}^n x_j\Big) \delta^2\Big(\sum_{j=1}^n x_j \vec{r}_{\perp j}\Big) \delta(x - x_a) \\ \times \delta^2\Big[\vec{b}_\perp + (1-x)\vec{r}_{\perp a} - \sum_{j \neq a}^n x_j \vec{r}_{\perp j}\Big] \\ \times \Psi_{n,c}^*(x_1, \ldots; \vec{r}_{\perp 1}, \ldots) \Psi_{n,c}(x_1, \ldots; \vec{r}_{\perp 1}, \ldots), \end{aligned}$$

where a denotes the index of the active parton, n is the number of partons in the Fock state, and the sum over c represents the sum over all additional quantum numbers characterizing the Fock state. Here, one explicitly observes $\lim_{x \to 1} q(x, \vec{b}_\perp) \propto \delta^2(\vec{b}_\perp)$. Since $H(x, 0, t)$ is the Fourier transform of the transverse distribution, the slope in $-t = \vec{\Delta}_\perp^2$ at the origin measures the rms transverse radius. As a result, we expect the substantial change in transverse size with x to be reflected in an equally significant change in slope with x. In particular, as $x \to 1$ the slope should approach zero. Hence, when we calculate moments of $H(x, 0, t)$, the higher the power of x, the more strongly large x is weighted, and the smaller the slope should become. Therefore, this argument makes the qualitative prediction that the slope of the generalized form factors $A_{n0}(t)$ and $\tilde{A}_{n0}(t)$ should decrease with increasing n, and we expect that this effect should be strong enough to be clearly visible in lattice calculations of these form factors.

III. Lattice measurement.—Generalized form factors $A_{(n=1,2,3),0}(t)$ and $\tilde{A}_{(n=1,2,3),0}(t)$ were calculated from the lattice matrix elements $\langle P' | \mathcal{O}^{\mu_1} | P \rangle, \langle P' | \mathcal{O}^{\{\mu_1\mu_2\}} | P \rangle$ and $\langle P' | \mathcal{O}^{\{\mu_1\mu_2\mu_3\}} | P \rangle$, and their spin-dependent counterparts using the new method introduced in Ref. [10]. We considered all the combinations of $\vec{P}$ and $\vec{P}'$ that produce the same four-momentum transfer $t = (P' - P)^2$ subject to the conditions that $\vec{P} = \frac{2\pi}{aN_s}(n_x, n_y, n_z)$ and $\vec{P}' = (0, 0, 0)$ or $\frac{2\pi}{aN_s}(-1, 0, 0)$. Using all these momentum combinations for a given t below 3.5 GeV^2, we calculated all the hypercubic lattice operators and index combinations that produce the same continuum GFFs, obtaining an overdetermined set of equations from which we extracted a statistically accurate measurement. The errors are substantially smaller than obtained by the common practice of measuring a single operator with a single momentum combination. Our calculations are based on unquenched SESAM configurations [11] on $16^3 \times 32$ lattices with $\kappa = 0.1560$ and $\kappa = 0.1570$, corresponding to pion masses of $m_\pi = 897$ and 744 MeV, respectively. As in Ref. [10], the one-loop perturbative renormalization factors of Ref. [12] were used to relate the lattice results to the continuum modified minimal subtraction ($\overline{\rm MS}$) scheme at the scale $\mu^2 = 4$ GeV^2.

Figure 1 presents our principal results, showing the generalized form factors $A_{n0}(t)$ and $\tilde{A}_{n0}(t)$ for the lowest three moments: $n = 1, 2$, and 3. The form factors have been normalized to unity at $t = 0$ to make the dependence of the shape on n more obvious. Note that $A_{1,0}$, $A_{3,0}$, and $\tilde{A}_{2,0}$ depend on the difference between the quark and antiquark distributions whereas $\tilde{A}_{1,0}$, $\tilde{A}_{3,0}$, and $A_{2,0}$ depend on the sum. Hence, only moments differing by two compare the same physical quantity with different weighting in x. To facilitate determination of the slope of the form factors and to guide the eye, the data have been fit using a dipole form factor

$$A_{n0}^{\text{dipole}} = \frac{A}{(1 - \frac{t}{m_d^2})^2}. \tag{8}$$

The solid line denotes the least-squares fit and the shaded error band shows the error arising from the statistical error in the fit mass, Δm_d. Although the dipole fit is purely phenomenological, we note that it is consistent with the lattice data. For reference, the normalization factors A_{n0} and dipole masses are tabulated in Table I.

The top panel in Fig. 1 shows the flavor nonsinglet case $A^u - A^d$, for which the connected diagrams we have calculated yield the complete answer. It is calculated at the heaviest quark mass we have considered, corresponding to $m_\pi = 897$ MeV. Note that the form factors are statistically very well separated, and differ dramatically for the three moments. Indeed, the slope at the origin decreases by more than a factor of 2 between $n = 1$ and $n = 3$, indicating that the transverse size decreases by more than a factor of 2. The second panel shows analogous results for lighter quarks, $m_\pi = 744$ MeV, where we observe the same qualitative behavior but slightly weaker dependence on the moment. The third panel shows the flavor singlet combination $A^u + A^d$, for which we have had to omit the disconnected diagram because of its significantly greater computational cost. Comparing this figure with the top panel calculated at the same quark mass, we observe that while the connected contributions to $A^u \pm A^d$ are qualitatively similar, there is significant quark flavor dependence that can be used to explore the nucleon wave function. It is useful to note our results for the u and d GFFs are consistent with the $n = 2$ moments calculated in Ref. [13]. The bottom panel shows the spin-dependent flavor nonsinglet form factors $\tilde{A}^u - \tilde{A}^d$ at the heaviest quark mass. Thus, comparing the top and bottom figures displays the difference between the spin-averaged and spin-dependent densities. We observe a striking difference, in that the change between the $n = 1$ and $n = 3$ form factors for $q(x, \vec{b}_\perp)_\uparrow - q(x, \vec{b}_\perp)_\downarrow$ is roughly 6 times smaller than for $\frac{1}{2}[q(x, \vec{b}_\perp)_\uparrow + q(x, \vec{b}_\perp)_\downarrow]$.

Finally, it is useful to use the slope of the form factors at $t = 0$ to determine the transverse rms radius,

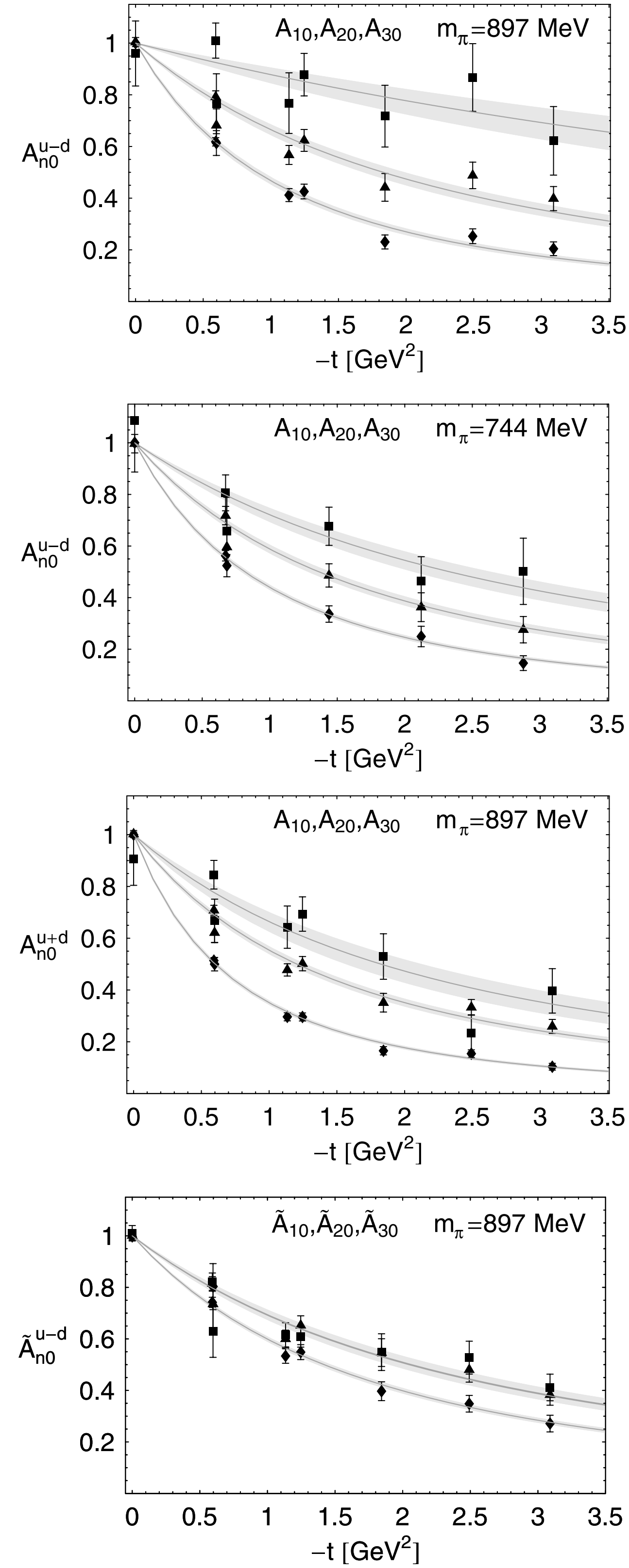

FIG. 1. Normalized lattice results for generalized form factors A_{n0} and $\tilde{A}_{n0}$ as a function of momentum transfer squared, $-t$, for $n = 1$ (diamonds), 2 (triangles), and 3 (squares). Note that disconnected diagram contributions are omitted from A^{u+d}.

TABLE I. Generalized form factors at $t = 0$, dipole masses, and transverse rms radii for the cases plotted in Fig. 1. Note that disconnected contributions are omitted from A^{u+d}.

GFF	$A(0)$	m_d (GeV)	$\langle r_\perp^2 \rangle^{1/2}$ (fm)
	$m_\pi = 897$ MeV ($\kappa = 0.1560$)		
$A_{1,0}^{u-d}(0)$	$1.000 \pm .001$	$1.470 \pm .031$	$0.380 \pm .008$
$A_{2,0}^{u-d}(0)$	$0.241 \pm .004$	$2.102 \pm .081$	$0.266 \pm .010$
$A_{3,0}^{u-d}(0)$	$0.060 \pm .008$	$3.857 \pm .494$	$0.145 \pm .019$
$A_{1,0}^{u+d}(0)$	$2.998 \pm .002$	$1.205 \pm .014$	$0.463 \pm .005$
$A_{2,0}^{u+d}(0)$	$0.666 \pm .009$	$1.706 \pm .040$	$0.327 \pm .008$
$A_{3,0}^{u+d}(0)$	$0.155 \pm .018$	$2.099 \pm .153$	$0.266 \pm .019$
$\tilde{A}_{1,0}^{u-d}(0)$	$1.195 \pm .014$	$1.850 \pm .028$	$0.302 \pm .005$
$\tilde{A}_{2,0}^{u-d}(0)$	$0.293 \pm .006$	$2.223 \pm .058$	$0.251 \pm .007$
$\tilde{A}_{3,0}^{u-d}(0)$	$0.123 \pm .004$	$2.233 \pm .087$	$0.250 \pm .010$
	$m_\pi = 744$ MeV ($\kappa = 0.1570$)		
$A_{1,0}^{u-d}(0)$	$1.001 \pm .001$	$1.402 \pm .019$	$0.398 \pm .005$
$A_{2,0}^{u-d}(0)$	$0.261 \pm .009$	$1.814 \pm .049$	$0.308 \pm .008$
$A_{3,0}^{u-d}(0)$	$0.071 \pm .013$	$2.373 \pm .138$	$0.235 \pm .014$

$$\langle r_\perp^2 \rangle^{(n)} = \frac{\int d^2 b_\perp b_\perp^2 \int dx x^{n-1} q(x, \vec{b}_\perp)}{\int d^2 b_\perp \int dx x^{n-1} q(x, \vec{b}_\perp)}. \tag{9}$$

Transverse rms radii calculated in this way are tabulated in Table I. To set the scale, the transverse charge radius at this mass is $\langle r_\perp^2 \rangle_{\rm charge} = 0.48$ fm, which is two-thirds the experimental transverse size 0.72 fm, reflecting the absence of a significant pion cloud. For the heaviest quark mass, $m_\pi = 897$ MeV, the nonsinglet transverse size $\langle r_\perp^2 \rangle_{\rm u-d} = 0.38$ fm is slightly smaller than the rms charge radius, but drops 62% to 0.14 fm for $n = 3$. The singlet size $\langle r_\perp^2 \rangle_{\rm u+d}$ is 0.46 fm and drops 43% to 0.27 for $n = 3$. This is a truly dramatic change in rms radius arising from changing the weighting by x^2. An alternative way to describe the same effect is in terms of the mean value of x. The mean value of x in the distribution $q(x)$ is of the order of 0.2 and roughly 0.4 in the distribution $x^2 q(x)$. In these terms, the nonsinglet transverse size drops 62% as the mean value of x increases from 0.2 to 0.4, and goes to zero when x reaches 1.

IV. Conclusions and outlook.—In the "heavy pion world" presently accessible to full lattice QCD, we have calculated the connected contributions to the lowest three generalized form factors A_{n0} and $\tilde{A}_{n0}$ up to $|t| = 3$ GeV as shown in Fig. 1. We obtain excellent precision for $n = 1$ and sufficient precision for $n = 2$ and 3 to clearly distinguish each form factor and observe striking differences in slope and hence transverse size. Whereas there are other calculations of isolated moments, three moments are crucial for the present investigation since $n = 1$ and 3 are necessary to measure the same combination of quark and antiquark distributions. The dependence of the transverse size on x is most dramatic for the heaviest $u - d$ combination, for which $\langle r_\perp^2 \rangle_{\rm u-d}$ decreases by 62% between the first and third moment. We also observed clear dependence of the transverse distribution on flavor and spin. Our results show that the commonly used factorization ansatz $H(x, 0, t) = Q(x)F(t)$ is fundamentally wrong in the heavy pion world and we are aware of no arguments as to why it should be restored for lighter quarks.

The most immediate challenges are to extend these calculations to the chiral regime of realistic quark masses, which is being explored using a hybrid calculation of dynamical staggered sea quarks and domain wall valence quarks [14], and to extend techniques for evaluating disconnected diagrams [15] to GFFs. When precise, controlled extrapolations to the physical pion mass are finally achieved, moments calculated from first principles will play an essential role in complementing experimental results because of the impracticality of measuring the full x, ξ, and t dependence of $H(x, \xi, t)$ and $\tilde{H}(x, \xi, t)$ experimentally. In addition, they will provide rich insight into the flavor and spin dependence of the transverse wave function.

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