Bell-Klyshko Inequalities to Characterize Maximally Entangled States of n Qubits

Zeqian Chen

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O.Box 71010, 30 West District, Xiao-Hong Mountain, Wuhan 430071, China

(Received 26 February 2004; published 7 September 2004)

This Letter presents the first rigorous proof of the conjecture raised by Gisin and Bechmann-Pasquinucci [Phys. Lett. A **246**, 1 (1998)], that the Greenberger-Horne-Zeilinger states of n qubits and the states obtained from them by local unitary transformations are the unique states that maximally violate the Bell-Klyshko inequalities. The proof is obtained by using the certain algebraic properties that Pauli's matrices satisfy and some subtle mathematical techniques. Since all states obtained by local unitary transformations of a maximally entangled state are equally valid entangled states, we thus give a characterization of maximally entangled states of n qubits in terms of the Bell-type inequality.

DOI: 10.1103/PhysRevLett.93.110403 PACS numbers: 03.65.Ud, 03.67.Mn

The Bell inequality [1] was originally designed to rule out various kinds of local hidden variable theories. Precisely, the Bell inequality indicates that certain statistical correlations predicted by quantum mechanics for measurements on two-qubit ensembles cannot be understood within a realistic picture based on Einstein, Podolsky, and Rosen's (EPR's) notion of local realism [2]. However, this inequality also provides a test to distinguish entangled from nonentangled quantum states of n qubits [3–5]. Moreover, in this Letter we will show that the Bell inequality presents a characterization of maximally entangled states of n qubits.

As is well known, maximally entangled states, such as Bell states and Greenberger-Horne-Zeilinger (GHZ) states [6], have become a key concept in the nowadays quantum mechanics. On the other hand, from a practical point of view, maximally entangled states have found numerous applications in quantum information [7]. A natural question is then how to characterize maximally entangled states. There are extensive earlier works on maximally entangled states [8], however, this problem is far from being completely understood today.

It is well known that the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality [9] nicely characterizes the two-qubit Bell states. For the case of n qubits, it is argued that maximally entangled states should maximally violate the Bell inequality [10]. Therefore, for characterizing maximally entangled states of n qubits, it is suitable to study the states that maximally violate the Bell inequality. This criterion is not precise, because there are infinitely many versions of the Bell inequality [11]. A natural generalization of the Bell-CHSH inequality for n qubits, called Bell-Klyshko inequalities, was presented by Klyshko and Belinskii [12] and Gisin and Bechmann-Pasquinucci [10]. The Bell-Klyshko inequality of n qubits is maximally violated by the GHZ states. Also, all states obtained by local unitary transformations of them maximally violate the Bell-Klyshko inequality and are equally valid entangled states [13]. This leads Gisin and Bechmann-Pasquinucci [10] to conjecture that the converse holds true, i.e., all states of n qubits maximally violating the Bell-Klyshko inequality are exactly the GHZ states and the states obtained from them by local unitary transformations, and consequently the Bell-Klyshko inequality characterizes the maximally entangled states of n qubits.

In this Letter, we answer affirmatively Gisin and Bechmann-Pasquinucci's conjecture. The techniques involved here are based on the determination of local spin observables of the associated Bell operator, which was recently introduced by the author [14]. We show that a Bell operator presents a maximal violation on a state if and only if the associated local spin observables satisfy the certain algebraic identities that Pauli's matrices satisfy. By using some subtle mathematical techniques, we can find those states that show maximal violation, which are exactly the states obtained from the GHZ states by local unitary transformations.

Let us give a brief review of the Bell-Klyshko inequality. The Bell-Klyshko inequality of n qubits is defined recursively ($n \geq 2$). Let A_j, A_j' denote spin observables on the jth qubit, $j = 1, \ldots, n$. Denote by $\mathcal{B}_1 = A_1$ and $\mathcal{B}_1' = A_1'$. Define

$$\mathcal{B}_n = \mathcal{B}_{n-1} \otimes \frac{1}{2}(A_n + A_n') + \mathcal{B}_{n-1}' \otimes \frac{1}{2}(A_n - A_n'), \quad (1)$$

$$\mathcal{B}_n' = \mathcal{B}_{n-1}' \otimes \frac{1}{2}(A_n + A_n') - \mathcal{B}_{n-1} \otimes \frac{1}{2}(A_n - A_n'). \quad (2)$$

Clearly, $\mathcal{B}_n'$ denotes the same expression $\mathcal{B}_n$ but with all the A_j and A_j' exchanged. We call $\mathcal{B}_n$ the Bell operator of n qubits [10]. Assuming "local realism" [2], one concludes the Bell-Klyshko inequality of n qubits as follows [10],

$$\langle \mathcal{B}_n \rangle \leq 1. \quad (3)$$

Note that the right-hand side of Eq. (3) is one but not two as in [10]; because of that, we define the Bell operator $\mathcal{B}_n$

recursively on the first-qubit spin observables, not from the usual two-qubit Bell operator. Indeed, the two-qubit Bell operator involved here is that

$$\mathcal{B}_2 = \frac{1}{2}(A_1 \otimes A_2 + A_1 \otimes A_2' + A_1' \otimes A_2 - A_1' \otimes A_2').$$

As shown in [10] (see below also), for $n \geq 2$,

$$\| \mathcal{B}_n \| \leq 2^{(n-1)/2}. \tag{4}$$

The main result we shall prove is that

Theorem: *A state $|\varphi\rangle$ of n qubits maximally violates Eq.*(3), *that is,*

$$\langle \varphi | \mathcal{B}_n | \varphi \rangle = 2^{(n-1)/2}, \tag{5}$$

if and only if it can be obtained by a local unitary transformation of the GHZ state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0 \cdots 0\rangle + |1 \cdots 1\rangle)$, i.e.,

$$|\varphi\rangle = U_1 \otimes \cdots \otimes U_n |\mathrm{GHZ}\rangle \tag{6}$$

for some n unitary operators $U_1, \ldots, U_n$ on $\mathbf{C}^2$.

Let us fix some notation. For $A_j^{(l)} = \vec{a}_j^{(l)} \cdot \vec{\sigma}_j$ $(1 \leq j \leq n)$, we write

$$(A_j, A_j') = (\vec{a}_j, \vec{a}_j'), \qquad A_j \times A_j' = (\vec{a}_j \times \vec{a}_j') \cdot \vec{\sigma}_j.$$

Here, $\vec{\sigma}_j$ is the Pauli matrices for the jth qubit; the norms of real vectors $\vec{a}_j^{(l)}$ in $\mathbf{R}^3$ are equal to 1. It is easy to check that

$$A_j A_j' = (A_j, A_j') + i A_j \times A_j', \tag{7}$$

$$A_j' A_j = (A_j, A_j') - i A_j \times A_j', \tag{8}$$

$$\|A_j \times A_j'\|^2 = 1 - (A_j, A_j')^2. \tag{9}$$

The sufficiency of *Theorem* is clear. Indeed, as noted in [10], the GHZ state $|\mathrm{GHZ}\rangle$ satisfies Eq. (5) with $A_j = \vec{a}_j \cdot \vec{\sigma}_j$ for $\vec{a}_j$ regularly distributed in the $x - y$ plane with angles $(j-1)(-1)^{n+1}(\pi/2n)$ with respect to the x axis and $\vec{a}_j' \perp \vec{a}_j$. For generic states $|\varphi\rangle$ of the form Eq. (6), they maximally violate the Bell-Klyshko inequality of n qubits with $U_j^* A_j U_j$ and $U_j^* A_j' U_j$, where A_j and A_j' are associated with $|\mathrm{GHZ}\rangle$ as above.

It remains to prove the necessity. The proof will be done by induction based on the two-qubit case, which was proved by the author [14]. Here, we outline the proof of the two-qubit case. By Eqs. (7) and (8), one has that

$$\mathcal{B}_2^2 = 1 + (A_1 \times A_1') \otimes (A_2 \times A_2'), \tag{10}$$

and so by Eq. (9), $\mathcal{B}_2^2 \leq 2$ and $\|\mathcal{B}_2^2\| = 2$ if and only if

$$(A_1, A_1') = (A_2, A_2') = 0. \tag{11}$$

In particular, if a two-qubit state $|\psi\rangle$ satisfies

$$\langle \psi | \mathcal{B}_2 | \psi \rangle = \sqrt{2}, \tag{12}$$

then Eq. (11) holds true and so $\{A_j, A_j', A_j''\}$ satisfies the algebraic identities that Pauli's matrices satisfy [15], i.e.,

$$A_j A_j' = -A_j' A_j = i A_j'', \tag{13}$$

$$A_j' A_j'' = -A_j'' A_j' = i A_j, \tag{14}$$

$$A_j'' A_j = -A_j A_j'' = i A_j', \tag{15}$$

$$A_j^2 = (A_j')^2 = (A_j'')^2 = 1, \tag{16}$$

where $A_j'' = A_j \times A_j'$ $(j = 1, 2)$. Therefore, by choosing A_j''-representation $\{|0\rangle_j, |1\rangle_j\}$, i.e.,

$$A_j''|0\rangle_j = |0\rangle_j, A_j''|1\rangle_j = -|1\rangle_j, \tag{17}$$

we have that

$$A_j|0\rangle_j = e^{-i\alpha_j}|1\rangle_j, A_j|1\rangle_j = e^{i\alpha_j}|0\rangle_j, \tag{18}$$

$$A_j'|0\rangle_j = i e^{-i\alpha_j}|1\rangle_j, A_j'|1\rangle_j = -i e^{i\alpha_j}|0\rangle_j, \tag{19}$$

for some $0 \leq \alpha_j \leq 2\pi$, $j = 1, 2$.

We write $|00\rangle_{12}$, etc., as shorthand for $|0\rangle_1 \otimes |0\rangle_2$. Then, $|\psi\rangle = \lambda_{00}|00\rangle_{12} + \lambda_{01}|01\rangle_{12} + \lambda_{10}|10\rangle_{12} + \lambda_{11}|11\rangle_{12}$, where $|\lambda_{00}|^2 + |\lambda_{01}|^2 + |\lambda_{10}|^2 + |\lambda_{11}|^2 = 1$. By using Eqs. (18) and (19) we conclude from Eq. (12) that $|\psi\rangle = \frac{1}{\sqrt{2}}(e^{i\phi}|00\rangle_{12} + e^{i\theta}|11\rangle_{12})$ for some $0 \leq \phi, \theta \leq 2\pi$. This immediately concludes that $|\psi\rangle = (U_1 \otimes U_2)\frac{1}{\sqrt{2}} \times (|00\rangle + |11\rangle)$ for unitary operators

$$U_1 = V_1\begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = V_2\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \tag{20}$$

where V_j is the unitary transform from the original σ_z^j representation to A_j'' representation on the jth qubit, i.e., $V_j|0\rangle = |0\rangle_j$ and $V_j|1\rangle = |1\rangle_j$ for $j = 1, 2$.

As follows, we write $A_1 A_n$, etc., as shorthand for $A_1 \otimes I \otimes \cdots I \otimes A_n$, $\mathcal{B}_{n-1}^2 = \mathcal{B}_{n-1}^2 \otimes I$, and $\mathcal{B}_j^2 A_{j+1}'' A_n'' = \mathcal{B}_j^2 \otimes A_{j+1}'' \otimes I \otimes \cdots I \otimes A_n''$, where $A_j'' = A_j \times A_j'$ $(1 \leq j \leq n)$, and I is the identity on a qubit. The proof of *Theorem* for necessity consists of the following five steps:

Proof: (i) At first, we show that

$$\mathcal{B}_n^2 = (\mathcal{B}_n')^2. \tag{21}$$

To this end, by using Eqs. (7) and (8) we conclude from Eqs. (1) and (2) that

$$\begin{aligned} \mathcal{B}_n^2 \quad = \quad & \tfrac{1}{2}[1 + (A_n, A_n')]\mathcal{B}_{n-1}^2 \\ & + \tfrac{1}{2}[1 - (A_n, A_n')](\mathcal{B}_{n-1}')^2 \\ & - [\mathcal{B}_{n-1}, \mathcal{B}_{n-1}'] \otimes (\tfrac{1}{2} i A_n''), \end{aligned}$$

and

$$\begin{aligned} (\mathcal{B}_n')^2 \quad = \quad & \tfrac{1}{2}[1 + (A_n, A_n')](\mathcal{B}_{n-1}')^2 \\ & + \tfrac{1}{2}[1 - (A_n, A_n')]\mathcal{B}_{n-1}^2 \\ & - [\mathcal{B}_{n-1}, \mathcal{B}_{n-1}'] \otimes (\tfrac{1}{2} i A_n''), \end{aligned}$$

respectively. Since $\mathcal{B}_1^2 = A_1^2 = I = (A_1')^2 = (\mathcal{B}_1')^2$, by induction, we conclude that

$$\mathcal{B}_n^2 = (\mathcal{B}_n')^2 = \mathcal{B}_{n-1}^2 - [\mathcal{B}_{n-1}, \mathcal{B}_{n-1}'] \otimes \left(\frac{1}{2} i A_n''\right). \quad (22)$$

(ii) Second, we prove that for $n \geq 3$,

$$\mathcal{B}_n^2 = \mathcal{B}_{n-1}^2 + A_1'' A_n'' + \sum_{j=1}^{n-2} \mathcal{B}_j^2 A_{j+1}'' A_n''. \quad (23)$$

We need compute $[\mathcal{B}_n, \mathcal{B}_n']$. From Eqs. (1) and (2) it concludes that

$$\begin{aligned}\mathcal{B}_n\mathcal{B}_n' \;=\;& i\mathcal{B}_{n-1}^2 \otimes A_n'' + \tfrac{1}{2}[1 + (A_n, A_n')]\mathcal{B}_{n-1}\mathcal{B}_{n-1}' \\ & - \tfrac{1}{2}[1 - (A_n, A_n')]\mathcal{B}_{n-1}'\mathcal{B}_{n-1},\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_n'\mathcal{B}_n \;=\;& -i\mathcal{B}_{n-1}^2 \otimes A_n'' + \tfrac{1}{2}[1 + (A_n, A_n')]\mathcal{B}_{n-1}'\mathcal{B}_{n-1} \\ & - \tfrac{1}{2}[1 - (A_n, A_n')]\mathcal{B}_{n-1}\mathcal{B}_{n-1}',\end{aligned}$$

respectively. Thus, we have that

$$[\mathcal{B}_n, \mathcal{B}_n'] = 2i\mathcal{B}_{n-1}^2 \otimes A_n'' + [\mathcal{B}_{n-1}, \mathcal{B}_{n-1}'] \otimes I$$

and recursively,

$$[\mathcal{B}_n, \mathcal{B}_n'] = 2i\left(A_1'' + \sum_{j=1}^{n-1} \mathcal{B}_j^2 A_{j+1}''\right). \quad (24)$$

Hence, by Eqs. (22) and (24) we obtain Eq. (23).

(iii) In the third step, we prove that $\|\mathcal{B}_n^2\| \leq 2^{n-1}$ and for $n \geq 2$, the equality holds if and only if

$$(A_j, A_j') = 0 \quad (25)$$

for all $j = 1, \ldots, n$.

By induction, assuming $\|\mathcal{B}_j^2\| \leq 2^{j-1}$ for $2 \leq j \leq n-1$, we conclude from Eq. (23) that

$$\begin{aligned}\|\mathcal{B}_n^2\| &\leq \|\mathcal{B}_{n-1}^2\| + 1 + \sum_{j=1}^{n-2} \|\mathcal{B}_j^2\| \leq 1 + \sum_{j=1}^{n-1} 2^{j-1} \\ &= 2^{n-1},\end{aligned}$$

noting that $\|\mathcal{B}_2^2\| \leq 2$ has been shown above.

We have known that $\|\mathcal{B}_2^2\| = 2$ if and only if $(A_1, A_1') = (A_2, A_2') = 0$. If $\|\mathcal{B}_n^2\| = 2^{n-1}$, by Eq. (23) we follow that both $\|\mathcal{B}_{n-1}^2\| = 2^{n-2}$ and $\|A_n''\| = 1$. By the inductive assumption, we conclude that $(A_j, A_j') = 0$ for all $j = 1, \ldots, n-1$. Also, by Eq. (9) we have that $(A_n, A_n') = 0$.

Conversely, assuming Eq. (25), we have that all $\{A_j, A_j', A_j''\}$ $(1 \leq j \leq n)$ satisfy Eqs. (13)–(16). We choose A_j''-representation $\{|0\rangle_j, |1\rangle_j\}$ on the jth qubit (e.g., Eq. (17)) and write $|0\cdots 0\rangle_n$, etc., as shorthand for $|0\rangle_1 \otimes \cdots \otimes |0\rangle_n$. By induction, we follow from Eq. (23) that $\mathcal{B}_n^2|0\cdots 0\rangle_n = 2^{n-1}|0\cdots 0\rangle_n$ and hence $\|\mathcal{B}_n^2\| = 2^{n-1}$.

(iv) In the fourth step, we prove that for $n \geq 2$, a state $|\varphi\rangle$ of n qubits satisfying

$$\mathcal{B}_n^2|\varphi\rangle = 2^{n-1}|\varphi\rangle \quad (26)$$

must be of the form

$$|\varphi\rangle = a|0\cdots 0\rangle_n + b|1\cdots 1\rangle_n, \quad (27)$$

where $|a|^2 + |b|^2 = 1$.

By step (iii), $\{|\epsilon_1\cdots\epsilon_n\rangle_n : \epsilon_1, \ldots, \epsilon_n = 0, 1\}$ is a orthogonal basis of the n-qubit system. Hence, we can uniquely write $|\varphi\rangle = \sum_{\epsilon_1,\ldots,\epsilon_n=0,1} \lambda_{\epsilon_1\cdots\epsilon_n}|\epsilon_1\cdots\epsilon_n\rangle_n$, where $\sum|\lambda_{\epsilon_1\cdots\epsilon_n}|^2 = 1$. By using Eq. (23), it concludes from Eq. (26) that

$$\mathcal{B}_{n-1}^2 \otimes I|\varphi\rangle = 2^{n-2}|\varphi\rangle, \quad (28)$$

$$A_1''A_n''|\varphi\rangle = |\varphi\rangle. \quad (29)$$

Since $|\varphi\rangle = \alpha|\varphi_1\rangle_{n-1} \otimes |0\rangle_n + \beta|\varphi_2\rangle_{n-1} \otimes |1\rangle_n$, where $|\varphi_1\rangle_{n-1}$ and $|\varphi_2\rangle_{n-1}$ are both states on the first $n-1$ qubits and $\alpha = \left(\sum_{\epsilon_1,\ldots,\epsilon_{n-1}=0,1}|\lambda_{\epsilon_1\cdots\epsilon_{n-1}0}|^2\right)^{1/2}$, $\beta = \left(\sum_{\epsilon_1,\ldots,\epsilon_{n-1}=0,1}|\lambda_{\epsilon_1\cdots\epsilon_{n-1}1}|^2\right)^{1/2}$, it concludes from Eq. (28) that $\mathcal{B}_{n-1}^2|\varphi_j\rangle_{n-1} = 2^{n-2}|\varphi_j\rangle_{n-1}$ for $j = 1, 2$. By the inductive assumption one has that $|\varphi_j\rangle_{n-1} = a_j|0\cdots 0\rangle_{n-1} + b_j|1\cdots 1\rangle_{n-1}$ with $|a_j|^2 + |b_j|^2 = 1$ for $j = 1, 2$. Hence $|\varphi\rangle = \lambda_1|0\cdots 0\rangle_n + \lambda_2|0\cdots 01\rangle_n + \lambda_3|1\cdots 10\rangle_n + \lambda_4|1\cdots 1\rangle_n$, where $\lambda_1 = \alpha a_1$, $\lambda_2 = \beta a_2$, $\lambda_3 = \alpha b_1$, and $\lambda_4 = \beta b_2$. However, it concludes from Eq. (29) that $\lambda_2 = \lambda_3 = 0$. Therefore, $|\varphi\rangle$ is of the form Eq. (27).

(v) Finally, we prove that if a state $|\varphi\rangle$ of n qubits with $n \geq 2$ satisfies Eq. (5), then

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(e^{i\phi}|0\cdots 0\rangle_n + e^{i\theta}|1\cdots 1\rangle_n), \quad (30)$$

for some $0 \leq \phi, \theta \leq 2\pi$.

By Eq. (4), one concludes that Eq. (5) is equivalent to that $\mathcal{B}_n|\varphi\rangle = 2^{(n-1)/2}|\varphi\rangle$. In this case, $|\varphi\rangle$ satisfies Eq. (26) and hence is of the form Eq. (27). Then, by Eq. (1) one has that

$$\begin{aligned}\mathcal{B}_n|\varphi\rangle = &\frac{1}{2}ae^{-i\alpha_n}[(1+i)\mathcal{B}_{n-1} + (1-i)\mathcal{B}_{n-1}'] \\ &\times|0\cdots 0\rangle_{n-1}|1\rangle_n + \frac{1}{2}be^{i\alpha_n}[(1-i)\mathcal{B}_{n-1} \\ &+(1+i)\mathcal{B}_{n-1}']|1\cdots 1\rangle_{n-1}|0\rangle_n,\end{aligned}$$

where α_n appears in Eqs. (18) and (19) for the nth qubit. Consequently,

$$\begin{aligned}\tfrac{1}{2}be^{i\alpha_n}[(1-i)\mathcal{B}_{n-1} + (1+i)\mathcal{B}_{n-1}']|1\cdots 1\rangle_{n-1} \\ = 2^{(n-1)/2}a|0\cdots 0\rangle_{n-1},\end{aligned} \quad (31)$$

and

$$\begin{aligned}\tfrac{1}{2}ae^{-i\alpha_n}[(1+i)\mathcal{B}_{n-1} + (1-i)\mathcal{B}_{n-1}']|0\cdots 0\rangle_{n-1} \\ = 2^{(n-1)/2}b|1\cdots 1\rangle_{n-1}.\end{aligned} \quad (32)$$

By Eq. (31) we have that

$$\begin{aligned} 2^{(n-1)/2}|a| &\le \frac{1}{2}|b|(|1-i|\|\mathcal{B}_{n-1}\| + |1+i|\|\mathcal{B}'_{n-1}\|) \\ &\le |b|(\sqrt{2}\cdot 2^{(n-2)/2} + \sqrt{2}\cdot 2^{(n-2)/2} \\ &= |b|2^{(n-1)/2}, \end{aligned}$$

since $\|\mathcal{B}_{n-1}\|, \|\mathcal{B}'_{n-1}\| \le 2^{(n-2)/2}$ by step (iii). This concludes that $|a| \le |b|$. Similarly, by Eq. (32) we have that $|b| \le |a|$ and so $|a| = |b|$. Therefore, we have that $a = \frac{1}{\sqrt{2}}e^{i\phi}$, $b = \frac{1}{\sqrt{2}}e^{i\theta}$ for some $0 \le \phi, \theta \le 2\pi$.

Now, denote by V_j the unitary transform from the original σ_z^j representation to A_j'' representation on the jth qubit, i.e., $V_j|0\rangle = |0\rangle_j$ and $V_j|1\rangle = |1\rangle_j$, and define

$$U_1 = V_1\begin{pmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = V_2\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix},$$

and $U_j = V_j$ for $3 \le j \le n$. Then U_j are all unitary operators on $\mathbf{C}^2$ so that Eq. (6) holds true and the proof is complete.

In conclusion, the maximal violation of the Bell-Klyshko inequality of n qubits only occurs for the states obtained from the GHZ states by local unitary transformations, as conjectured by Gisin and Bechmann-Pasquinucci [10]. However, it is proved by Werner and Wolf [16] that the orbit corresponding to the Bell-Klyshko inequality is the only one for which the overall maximal violation of all-multipartite Bell-correlation inequalities for two dichotomic observables per site is attained. Hence, the overall maximal quantum violation of all Bell-correlation inequalities for n qubits also occurs only for the GHZ states and the states obtained from them by local unitary transformations. Note that the GHZ state leads to a conflict with EPR's local realism for nonstatistical predictions of quantum mechanics [6]. This concludes that the overall maximal violation of all Bell-correlation inequalities of n qubits implies 100% violation between quantum mechanics and EPR's local realism on n qubits (except for the two-qubit case [17]), although the Bell inequalities indicate that EPR's local realism is only in conflict with the statistical prediction of quantum mechanics. Therefore, from the view of EPR's local realism, the maximally entangled states of n qubits should be just GHZ states and the states obtained from them by local unitary transformations. Our result sheds some new light on the close relationship among the maximal violation of the Bell-type inequalities, 100% violation between quantum mechanics and EPR's local realism, and maximally entangled states of multipartite systems. It turns out that the Bell-type inequalities can be used to reveal what the term maximally entangled states should actually mean in the multipartite systems. We may expect such a result to hold for higher dimensional quantum systems beyond qubits.

Note added.—After the acceptance of the present work, I became aware that the result presented here can be obtained by using the (different) method involved in [18]. However, the authors do not state it as a single statement nor discuss the significance of the result associated with maximally entangled states.

*Electronic address: zqchen@wipm.ac.cn

[1] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).

[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[3] N. Gisin, Phys. Lett. A **154**, 201 (1991);D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. **88**, 170405 (2002).

[4] J. Uffink, Phys. Rev. Lett. **88**, 230406 (2002); M. Seevinck and G. Svetlichny, Phys. Rev. Lett. **89**, 060401 (2002); K. Nagata, M. Koashi, and N. Imoto, Phys. Rev. Lett. **89**, 260401 (2002).

[5] S.-X. Yu, Z.-B. Chen, J.-W. Pan, and Y.-D. Zhang, Phys. Rev. Lett. **90**, 080401 (2003).

[6] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69; N. D. Mermin, Phys. Today **43**, No. 6, 9 (1990); N. D. Mermin, Am. J. Phys. **58**, 731 (1990).

[7] N. Gisin, G. Ribordy, W. Tittle, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002); M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[8] A search for "maximally entangled states" in title on www.arxiv.org/quant-ph yields 36 results.

[9] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[10] N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A **246**, 1 (1998).

[11] References can be found in: A. Cabello, quant-ph/0012089.

[12] D. N. Klyshko, Phys. Lett. A **172**, 399 (1993); A. V. Belinskii and D. N. Klyshko, Phys. Usp. **36**, 653 (1993).

[13] N. Linden and S. Popescu, Fortschr. Phys. **46**, 567 (1998).

[14] Z.-Q. Chen, Phys. Rev. A 70, 024303 (2004).

[15] W. Pauli, Z. Phys., **43**, 601 (1927).

[16] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001); M. Żukowski and Č. Brukner, Phys. Rev. Lett., **88**, 210401 (2002).

[17] Z.-Q. Chen, Phys. Rev. A **68**, 052106 (2003).

[18] V. Scarani and N. Gisin, J. Phys. A: Math. Gen. **34**, 6043 (2001).