Aging Transition and Universal Scaling in Oscillator Networks

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Self-sustained oscillators may turn non-self-oscillatory as a result of some kind of deterioration, which we call *aging* for simplicity. We discuss the effect of aging on the behavior of globally and diffusively coupled oscillators which are either all periodic or chaotic. It is shown that at a certain level of aging, macroscopic oscillation stops in a way which depends on the coupling strength. A universal scaling function to describe it is analytically derived and numerically verified.

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Coupled oscillators play a crucial role in a variety of areas in science and technology. Their dynamics underlie many important activities of living organisms to maintain life, e.g., heart contraction, peristaltic motion of gastrointestinal tracts, circadian rhythms [1] and visual information processing in mammalian brains [2]. Studies of coupled oscillators may therefore provide keys to elucidate the nature of life. Moreover, such studies may be useful in technological applications, examples of which include designing central pattern generators [3] for the locomotion of robots and coupled Josephson junctions as a stable source of voltage [4]. Experimental studies are also in progress, checking theories developed so far [5].

Coupled-oscillator systems supporting life as mentioned above are composed of a large number of oscillatory elements [1]. This seems quite reasonable, because the macroscopic activity of a large-scale system should be robust against various damages or deterioration such that some elements turn non-self-oscillatory, which we shall call *aging* for simplicity. However, to the authors' knowledge, it has not yet been investigated how robust a population of coupled oscillators can be against such aging and in what way its macroscopic activity is lost when the deterioration proceeds. These questions are important not only in understanding the robustness of the function of diverse biological and physiological systems, but also in technological contexts where a central problem is to design a robust system.

In this Letter, we study the effect of aging on the macroscopic activity of globally and diffusively coupled oscillators by increasing the ratio of non-self-oscillatory elements p from zero [6]. As oscillators, which are assumed identical in this Letter, we deal with both limit-cycle oscillators and chaotic oscillators. By introducing an order parameter, we show that macroscopic oscillation stops at a critical value of p which depends on the coupling strength K. Moreover, it is shown that such a transition is characterized by a universal scaling law involving both p and K. Numerical integrations in this work were performed for random initial conditions by means of the fourth order Runge-Kutta method with time step 0.1.

We begin with an analysis of globally coupled Stuart-Landau equations of the form (see, e.g., [7–11])

$$\dot{z}_j = (\alpha_j + i\Omega - |z_j|^2)z_j + \frac{K}{N}\sum_{k=1}^{N}(z_k - z_j) \qquad (1)$$

for $j = 1, \ldots, N$, where the overdot means differentiation with respect to time t, z_j is the complex amplitude of the jth oscillator, α_j is a parameter specifying the distance from a Hopf bifurcation, Ω is the natural frequency, and K is the coupling strength. Without coupling, the jth element exhibits periodic oscillations if $\alpha_j > 0$, and settles down at the trivial fixed point $z_j = 0$ if $\alpha_j \leq 0$. We assume that aging of the system proceeds in such a way that an active oscillator with $\alpha_j = a > 0$ turns inactive with $\alpha_j = -b < 0$, where both a and b are parameters. For convenience, we set the group of active elements to $j \in \{1, \ldots, N(1-p)\} \equiv S_a$ and that of inactive elements to $j \in \{N(1-p)+1, \ldots, N\} \equiv S_i$. We suppose that the system size N is large enough to enable us to regard the ratio p virtually as a continuous variable. In this setting, the system with $p = 0$ and $K > 0$ falls in perfect synchronization, in which each element oscillates with amplitude $\sqrt{a}$ and frequency Ω.

We now check the effect of aging through the behavior of the order parameter $|Z|$, where $Z \equiv N^{-1}\sum_{j=1}^{N} z_j$. Figure 1 shows its normalized values against p for some values of K. It is found that the order parameter vanishes at a critical value of p, p_c. For $p \geq p_c$, the system falls into the trivial fixed point $z_1 = \ldots = z_N = 0$ and is no longer active. We propose to call such a transition an *aging transition.* Figure 1 shows that for decreasing K, p_c increases until it reaches unity at a threshold value of K, K_c, below which p_c remains at unity.

The pair of critical values p_c and K_c can be obtained by assuming that in each group, all elements are in an identical state. Setting $z_j = A$ for all active elements and $z_j = I$ for all inactive elements, we obtain

$$\dot{A} = (a - Kp + i\Omega - |A|^2)A + KpI, \qquad (2)$$

$$\dot{I} = (-b - Kq + i\Omega - |I|^2)I + KqA, \qquad (3)$$

where $q \equiv 1 - p$. Then, it follows that the aging transi-

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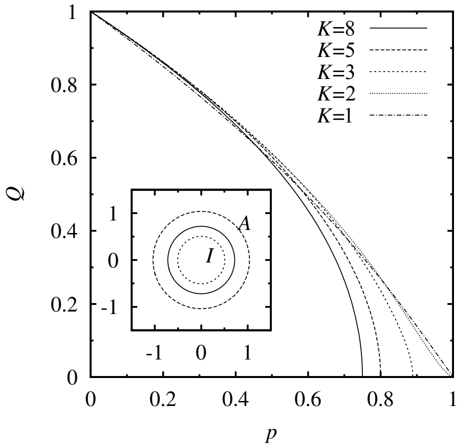

FIG. 1. Aging in the coupled Stuart-Landau equations, where $N = 1000, a = 2, b = 1, \Omega = 3$, and $Q \equiv |Z(p)|/|Z(0)|$. The inset shows trajectories of z_j for $K = 3, p = 0.6$ in the complex plane with the abscissa and ordinate meaning the real and imaginary part, respectively; both active (dashed curve) and inactive (dotted curve) elements are perfectly synchronized within each group; the solid curve shows the trajectory of Z.

tion occurs when the trivial fixed point $A = I = 0$ is stabilized as p is increased from zero. A linear stability analysis shows that

$$p_c = \frac{a(K + b)}{(a + b)K}, \tag{4}$$

below which a stable limit cycle bifurcates (see the inset of Fig. 1). This result implies that $K_c = a$, in accord with simulation. The reduction to the four-dimensional system (2) and (3) is validated by the fact that the attractors of the reduced system can be shown to be stable in the full system as well [12].

We found similar results for some other examples of globally and diffusively coupled limit-cycle oscillators. One of them is coupled Rössler equations [13] of the form

$$\dot{x}_j = -y_j - z_j + \frac{K}{N}\sum_{k=1}^{N}(x_k - x_j), \tag{5}$$

$$\dot{y}_j = x_j + c_j y_j + \frac{K}{N}\sum_{k=1}^{N}(y_k - y_j), \tag{6}$$

$$\dot{z}_j = d_j + z_j(x_j - e_j) + \frac{K}{N}\sum_{k=1}^{N}(z_k - z_j) \tag{7}$$

for $j = 1, \ldots, N$ (for a similar system, see [14]), where c_j, d_j, and e_j are all constants. As an example, we set $c_j = d_j = 0.2, e_j = 1 (j \in \mathrm{S_a}), c_j = d_j = -0.2, e_j = 2.5 (j \in \mathrm{S_i})$. For $K = 0$, each element of the first group exhibits periodic oscillations, while every member of the second group falls into a fixed point. As a measure of activity, we use the amplitude of macroscopic oscillation defined by

$$M = \sqrt{<(\mathbf{X}_c - <\mathbf{X}_c>)^2>}, \tag{8}$$

where $\mathbf{X}_c \equiv N^{-1}\sum_{j=1}^{N}(x_j, y_j, z_j)$ is the centroid and the bracket means a long time average. Note that for the coupled Stuart-Landau equations, M is equal to $|Z|$. The behavior of M in the coupled Rössler equations is shown in Fig. 2 for $N = 1000$ and some values of K, where we again find aging transitions. For large K, M shows a resonancelike behavior before it vanishes at $p = p_c$. Intriguingly, in such a case, the activity of the system is enhanced by increasing the ratio of inactive elements. Figure 3 shows where the transition occurs in the (K, p) plane; in the displayed area, for decreasing K, p_c first increases to reach unity and then remains there, just as in the coupled Stuart-Landau equations. As also shown, this simulation result is reproduced by a linear stability analysis of the fixed point of a six-dimensional system created by the same reduction procedure as for (1). However, this figure also indicates that such a reduction breaks down in some regions where K is small. In such a case, what is called clustering (see, e.g., [11,15]) was observed to happen in the active group [12,16].

The above results are not restricted to limit-cycle oscillators. Figure 4 is devoted to a case of chaotic oscillators, i.e., equations (5)–(7) with the same parameter values as above, except for $e_j = 5.7 (j \in \mathrm{S_a})$. Without coupling, each element of the active group exhibits chaos [13]. As we see, the behavior of M is similar to those displayed in Figs. 1 and 2. However, the bifurcation structure leading to an aging transition is richer, as demonstrated in the inset of Fig. 4. As p is increased, a reverse period-doubling cascade occurs, leading to a one-loop limit cycle, which then disappears at $p = p_c$, giving way to a stable fixed point. It was confirmed that the corresponding phase diagram is qualitatively similar to Fig. 3, though the six-dimensional reduction is violated in a much wider area within the region $K < K_c$ [12].

The aging transition is a critical phenomenon featured by the existence of two critical parameter values p_c and K_c. The behavior of the order parameter near $p = p_c$ and $K = K_c$ is therefore especially interesting. We expect that

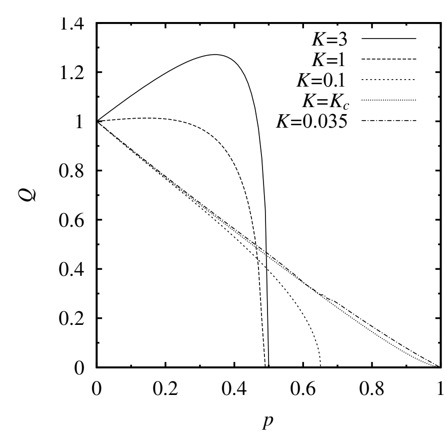

FIG. 2. Aging in the coupled periodic Rössler systems, where $N = 1000$, $Q \equiv M(p)/M(0)$, and $K_c = 0.046\,736\,55\ldots$.

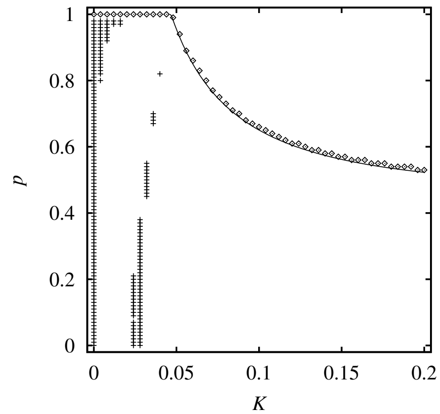

FIG. 3. Phase diagram for the same system as treated in Fig. 2 except for $N = 100$. The diamonds mark aging transition points, while the solid curve is the corresponding result obtained by reduction to a six-dimensional system. The crosses show where such a reduction breaks down in the sense that for at least one variable, the maximum and minimum values within a group after a sufficiently long time (2.1×10^4) differ by more than 10^{-4}. Note that the breakdown at $K = 0$ is trivial, because initial conditions are random.

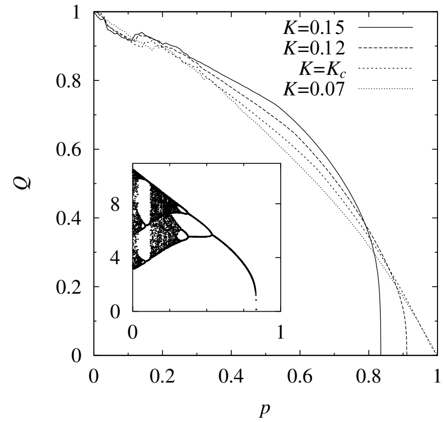

FIG. 4. Aging in the coupled chaotic Rössler systems, where $N = 1000$, $Q \equiv M(p)/M(0)$, and $K_c = 0.097\,177\,87\ldots$. The inset shows a bifurcation diagram of the centroid $\mathbf{X}_c \equiv (x_c, y_c, z_c)$ for $K = 0.15$ on a Poincaré surface defined by $x_c = 0$, $y_c < 0$; the abscissa is p, while the ordinate shows $\sqrt{y_c^2 + z_c^2}$ on the surface.

near $p = p_c$,

$$M \propto (p_c - p)^\beta \tag{9}$$

with an exponent $\beta > 0$. Indeed, for the coupled Stuart-Landau equations, it is easy to show analytically that $\beta = 1/2(K > K_c = a), 3/2(K = K_c)$, and $1(K < K_c)$. Curiously, the changes of β are not monotonic with respect to K. Actually, the same result seems to hold in all coupled-oscillator systems which were found to exhibit aging transitions. Figure 5 shows evidence in the case of coupled periodic Rössler systems studied above. We also observed crossover phenomena near $K = K_c$, i.e,. as p approaches p_c, β switches from 3/2 to either $1/2(K > K_c)$ or $1(K < K_c)$.

In order to explain these results, let us consider a general form of globally and diffusively coupled oscillators as follows:

$$\dot{\mathbf{x}}_j = \mathbf{F}_j(\mathbf{x}_j) + \frac{K}{N}\sum_{k=1}^{N} D \cdot (\mathbf{x}_k - \mathbf{x}_j) \tag{10}$$

for $j = 1, \ldots, N$, where $\mathbf{x}_j$ is the state vector of the jth element, and $\mathbf{F}_j = \mathbf{F}(j \in \mathrm{S_a}), \mathbf{G}(j \in \mathrm{S_i})$, and D is a constant diffusion matrix. Suppose that the dynamical system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is *active* in the sense that it exhibits a nonstationary behavior such as periodic oscillation and chaos, and that the dynamical system $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ is *inactive* in the sense that it falls into a stable fixed point, which is denoted by $\mathbf{I}_0$ below. We also consider the reduced system obtained by setting $\mathbf{x}_j = \mathbf{A}(j \in \mathrm{S_a})$, $\mathbf{I}(j \in \mathrm{S_i})$, i.e.,

$$\dot{\mathbf{A}} = \mathbf{F}(\mathbf{A}) + KpD \cdot (\mathbf{I} - \mathbf{A}), \tag{11}$$

$$\dot{\mathbf{I}} = \mathbf{G}(\mathbf{I}) + K(1 - p)D \cdot (\mathbf{A} - \mathbf{I}). \tag{12}$$

Below, we assume that this reduced system has a fixed point, say $(\mathbf{A}^*, \mathbf{I}^*)$, and that for a certain value of p, $\tilde{p}(K)$, it is stable for $p > \tilde{p}(K)$, but unstable for $p < \tilde{p}(K)$, giving way to a stable limit cycle through a supercritical Hopf bifurcation at $p = \tilde{p}(K)$. Our empirical results suggest that these assumptions hold fairly generally. Note that in the reduced system, p is a free parameter, which may be larger than unity. On the basis of the simulation results, we assume that $p_c = \tilde{p}(K) < 1(K > K_c)$, $p_c = \tilde{p}(K) = 1(K = K_c)$, and $p_c = 1$, $\tilde{p}(K) > 1(K < K_c)$. Note also that $\mathbf{I}^* = \mathbf{I}_0$ for $p = 1$. The critical coupling strength K_c is therefore determined by requiring that the fixed point $\mathbf{A}^*$ of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + K_c D \cdot (\mathbf{I}_0 - \mathbf{x})$ be marginally stable.

We now discuss the critical behavior of M. Inspired by simulation results, we assume that except a region where K is substantially smaller than K_c, the behavior of the

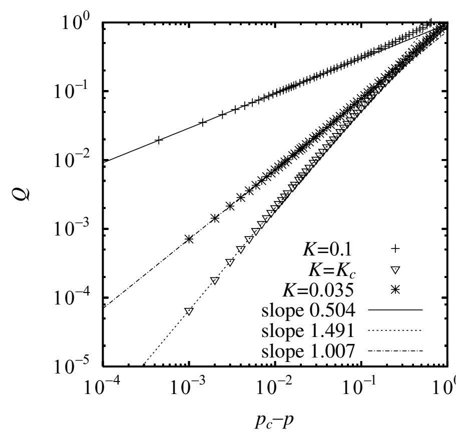

FIG. 5. Critical scaling of M in the coupled periodic Rössler systems with the same details as in Fig. 2. The slopes were computed by the method of least squares for the data in the range $p_c - p \leq 0.01$.

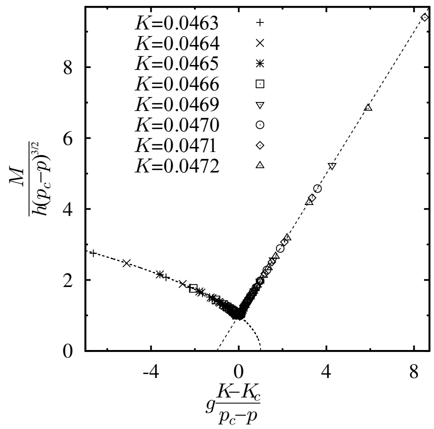

FIG. 6. Crossover scaling in the coupled periodic Rössler systems near both $K = K_c$ and $p = p_c$, where the details are the same as in Fig. 2. The dashed line and curve show the scaling function $\Phi(x)$ given in text. The parameters h and g were determined through the method of least squares.

original system (10) can be successfully reproduced by the reduction to (11) and (12). Then, the result $\beta = 1/2 (K > K_c)$ follows from the nature of Hopf bifurcation. The power law for $K < K_c$, $\beta = 1$, can be easily derived from the assumed stability of $\mathbf{I}_0$ [12]. We therefore focus on a close neighborhood of $K = K_c$. Let us express the centroid of the system (10) as $\mathbf{X}_c = (1-p)(\mathbf{A}^* + \delta\mathbf{A}) + p(\mathbf{I}^* + \delta\mathbf{I})$, where $\delta\mathbf{A} \equiv \mathbf{A} - \mathbf{A}^*$ and $\delta\mathbf{I} \equiv \mathbf{I} - \mathbf{I}^*$. The key observation is that for $0 < \tilde{p}(K) - p \ll 1$, we can put $\delta\mathbf{A} = [\tilde{p}(K) - p]^{1/2}\mathbf{a}$, $\delta\mathbf{I} = [\tilde{p}(K) - p]^{1/2}(1-p)\mathbf{i}$, where $\mathbf{a}$ and $\mathbf{i}$ are such quantities that the variance does not vanish at $p = 1$ and $p = \tilde{p}(K)$. This follows from the assumption that $p = \tilde{p}(K)$ is a supercritical Hopf bifurcation point in the reduced system and also that near $p = 1$, $\mathbf{I} - \mathbf{I}_0 = O(1-p)$ as well as $\mathbf{I}^* - \mathbf{I}_0 = O(1-p)$, as follows from the form of (12). Then, we find out that $\beta = 3/2$ for $K = K_c$.

The above analysis reveals a universal scaling law of the order parameter M near both $p = p_c$ and $K = K_c$. Introducing $g \equiv |\tilde{p}'(K_c)|$, where the prime means differentiation by K, and h denoting the standard deviation of $\mathbf{a} + p\mathbf{i}$ at $p = 1$ and $K = K_c$, we obtain

$$\frac{M}{h} = (p_c - p)^{3/2}\Phi\left(g\frac{K - K_c}{p_c - p}\right), \tag{13}$$

where $\Phi(x) = 1 + x \quad (x \geq 0), \sqrt{1-x} \quad (x < 0)$. This crossover scaling law is exemplified in Fig. 6. This result implies that the crossover occurs when $p_c - p \sim g|K - K_c|$, provided K is sufficiently close to K_c.

In summary, as the ratio of inactive elements p exceeds a certain value, p_c, globally and diffusively coupled oscillators lose their macroscopic activity. This aging transition is characterized by a universal scaling law of an order parameter concerning both p and the coupling strength K. Empirically, this seems to be a fairly general scenario, though further studies are necessary. The critical level of aging p_c is a measure of the robustness of macroscopic oscillation in the original system with $p = 0$. This work reveals that typically it decreases for increasing K. Therefore, although favorable for coherence at $p = 0$, large coupling strength may be harmful in view of the robustness against aging. It might be that coupling strengths in living tissues and organs are optimally tuned in this sense. A remaining subject is to examine the effect of aging on nonidentical oscillators, e.g., with distributed natural frequencies. Such a study will be reported elsewhere together with the details of this work [12].

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