Self-Stabilized Fractality of Seacoasts through Damped Erosion

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Erosion of rocky coasts spontaneously creates irregular seashores. But the geometrical irregularity, in turn, damps the sea waves, decreasing the average wave amplitude. There may then exist a mutual self-stabilization of the wave amplitude together with the irregular morphology of the coast. A simple model of such stabilization is studied. It leads, through a complex dynamics of the earth-sea interface, to the appearance of a stationary fractal seacoast with a dimension close to 4/3. Fractal geometry here plays the role of a morphological attractor directly related to percolation geometry.

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Coastline morphology is of current interest in geophysical research [1], and coastline erosion has important economic consequences [2]. Also, the recent concern about global warming has increased the demand for a better understanding of coastal evolution. At the same time, although the geometry of seacoasts is often used as an introductory archetype of fractal morphology in nature [3,4], there has been no explanation about which physical mechanism could justify that empirical observation. In the field literature [5] one can read, *"As a matter of some urgency, researchers concerned with coastal evolution should consider the alternative models, even if there are few supporting data. The ideas of… stochastic development,… and criticality, all deserve investigation."* The present work proposes a minimal, but robust, model of evolution of rocky coasts towards fractality.

The model describes how a stationary fractal geometry can arise spontaneously from the mutual self-stabilization of coast morphology and sea eroding power [6]. If, on one hand, erosion generally increases the geometrical irregularity of the coast, on the other hand this increase creates a stronger damping of the sea and a consequent diminution of its eroding power. The increased damping argument relies on the studies of fractal acoustical cavities, which have shown that viscous damping is augmented on a longer, irregular surface [7,8]. In the following, a minimal two-dimensional model of erosion is introduced which leads to the spontaneous evolution of a smooth seashore towards fractality as shown in Fig. 1. This fractality is then shown to be a property of the permanently changing shape of the coastline during long term erosion.

Rocky coast erosion is the product of marine and atmospheric causes [9]. There exist many different erosion processes: wave quarrying, abrasion, wetting and drying, frost shattering, thermal expansion, salt water corrosion, carbonation, and hydrolysis. A simplified picture is used here by assuming that the different processes can be separated into two categories: "rapid" mechanical erosion (namely, wave quarrying) and "slow" chemical weakening. The justification is that mechanical erosion generally occurs rapidly, mainly during storms, after rock has been altered and weakened by the slow weathering processes. We first study the supposedly rapid erosion mechanisms. Then we show that the full complex dynamics, involving fast and slow processes, build a dynamic equilibrium that changes the shape of the coast but preserves its fractal properties.

The sea, together with the coast, is considered to constitute a resonator. It is assumed that there exists an average excitation power of the waves P_0. The "force" acting on the unitary length of the coast is measured by the square of the wave amplitude Ψ^2. This wave amplitude is related to P_0 by a relation of the type $\Psi^2 \sim P_0 Q$, where Q is the morphology dependent quality factor of the system; the smaller the quality factor, the stronger the damping of the sea waves. There are several causes of wave damping. Since the different loss mechanisms occur independently, the quality factor satisfies a relation of the type

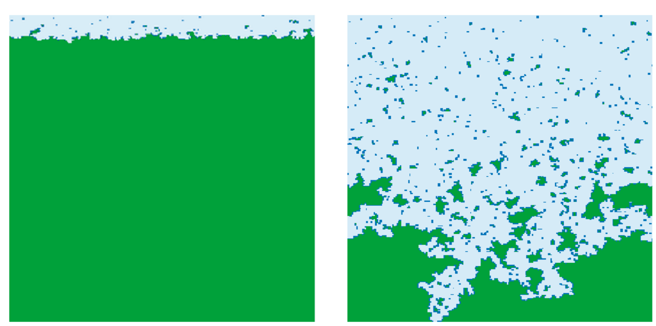

FIG. 1 (color). Time evolution of the coastline morphology. (Left) beginning of the erosion process: the coast is irregular but not yet fractal. (Right) coastline at the end of the rapid erosion process.

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$$\frac{1}{Q} = \frac{1}{Q_{\text{coast}}} + \frac{1}{Q_{\text{other}}}, \tag{1}$$

where Q_{coast} is the quality factor due to the viscous dissipation of the fluid moving along the coast and the nearby islands and Q_{other} is related to other damping mechanisms (e.g., bulk viscous damping). Studies of fractal acoustical cavities [7,8] have shown that the viscous damping increases roughly proportionally to the cavity perimeter. Therefore, one can, in first approximation, assume that Q_{coast} is inversely proportional to the coast perimeter $L_p(t)$, whereas Q_{other} is independent of the coast morphology. In other words, the sea exerts a homogeneous erosion force $f(t)$ on each coast element proportional to $\Psi^2(t)$:

$$f(t) = \frac{f_0}{1 + \frac{gL_p(t)}{L_0}}, \tag{2}$$

where $L_p(t)$ is the total length of the coast at time t [then $L_p(t = 0) = L_0$]. The factor g measures the relative contribution to damping of a flat shore as compared with the total damping. Since g represents the importance of coast dissipation with respect to the other mechanisms, which are thought to be dominant, computations are performed under the condition $g \ll 1$. The quantity f_0 is the renormalized value of P_0 such that $f(t) < 1$ at all t.

The mechano-chemical properties of the rocks constituting the coast, which are linked to structure and composition defining their "lithology," are unknown and exhibit some dispersion. The "resisting" random earth is then modeled by a square lattice of random units of width L_0. Each site represents a small portion of the earth. The sea acts on a shoreline constituted of cells, each one characterized by a random number x_i, between 0 and 1, representing its lithology. The erosion model should also take into account the fact that a site surrounded by the sea is relatively weakened as compared with a site surrounded by earth or other coast sites. Hence, the resistance to erosion r_i of a site depends on both its lithology and the number of sides exposed to the action of the sea. This is implemented through the following weakening rule: sites surrounded by three earth sites have a resistance $r_i = x_i$. If the contact is with two sea sites, the resistance is $r_i = x_i^2$. And, if site i is attacked by three or four sides, it has zero resistance. The iterative evolution rule is simple: at time t, all coast sites with $r_i < f(t)$ are eroded, and then $L_p(t)$ and $f(t)$ are updated together with the resistances of the earth sites in contact with the sea. Then, from one step to the next, some sites are eroded because they present a "weak lithology," while some strong sites are eroded due to their weaker stability due to sea neighboring. An example of local evolution is shown in Fig. 2.

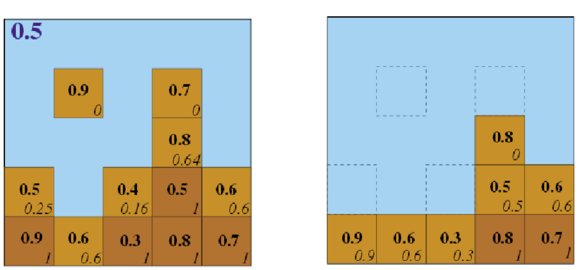

FIG. 2 (color). Illustration of the erosion process. The thick numbers in the square centers represent the lithology $\{x_i\}$. The numbers in the corners are the corresponding resistances r_i which depend on the local environment as explained in the text. The sites marked with 1 are earth sites with no contact with the sea. Left and right pictures represent, respectively, the situations before and after an erosion step with $f(t) = 0.5$. After this step resistances are updated due to the new sea environment.

In the first steps of the dynamics, the erosion front keeps quite smooth, and it roughens progressively as shown in Fig. 1. During the process, finite clusters are detached from the infinite earth, creating *islands*. At any time, both the islands and the coastline perimeters contribute to the damping. As the total coastline length $L_p(t)$ increases, the sea force becomes weaker. At a certain time step t_f, the weakest point of the coast is stronger than $f(t_f)$ and the rapid dynamics stops. This shows that erosion reinforces the coast by preferential elimination of its weakest elements until the coast is strong enough to resist further erosion. The time evolution of $f(t)$ is shown in Fig. 3, upper plot. The fact that the process spontaneously leads to a stop of erosion for a *nonzero* wave power is remarkable. It is a direct consequence of random percolation [10] as discussed below.

At time t_f the coastline is fractal (see Fig. 1) up to a characteristic width σ. This width σ is defined as the standard deviation of the final coastline depth

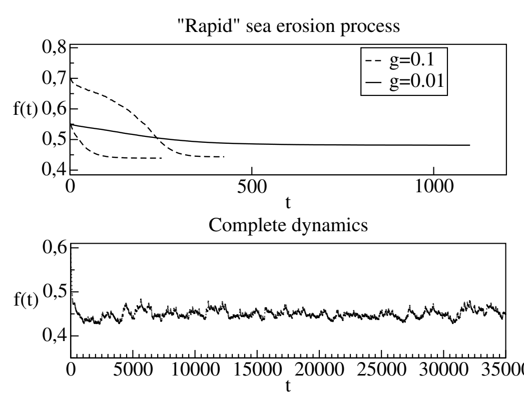

FIG. 3. Time dependence of the erosion force $f(t)$. The upper plot shows the evolution of the sea-erosion force acting on the coastline during a rapid sea-erosion process [different values of the scale gradients g and of $f(0)$]. This dynamics spontaneously stops at a value weakly dependent on g (systems with $L_0 = 1000$, averaged over ten different realizations). The lower plot is the erosion force $f(t)$ during the complete dynamics (slow weathering process triggering rapid erosion).

$\langle (y - Y)^2 \rangle^{1/2}$ (where $Y = \langle y \rangle$ is the average erosion depth). Figure 4 (left plot) shows the box-counting determination of the fractal dimension $D_f = 1.332(3)$, a value very close to 4/3. Such a fact indicates the close relation with percolation, as 4/3 is the dimension of the so-called accessible percolation cluster [11].

Moreover, a detailed study of the coast width σ indicates that the model pertains to the universality class of gradient percolation (GP) [12]. As shown in Fig. 3 (right plot), the coast width σ follows a scaling law with respect to g, $\sigma \sim g^{-\alpha_\sigma}$ with $\alpha_\sigma \simeq 4/7$. This law is characteristic of GP where sites are occupied at random with a probability that varies between 1 and 0 along one fixed direction with a constant gradient g, and σ is the width of the frontier of the infinite cluster. Here, g is proportional to a gradient of occupation probability from the following argument. At time t, the erosion power is $f(t)$ while the sea has eroded the earth up to an average depth $Y(t)$, an increasing function of t. Inverting this function, f can be written as $f(t(Y))$. There exists then a spatial gradient of the occupation probability by the sea. For small enough g one can write $|df/dY| = |(g/L_0)[dL_p(Y)/dY]|$. The quantity $dL_p(Y)/dY$ is a function of g, but to the lowest order it is a constant independent of g since even with $g = 0$, there will be an erosion due to randomness and a consequent perimeter evolution $L_p(t)$. Then to lowest order, the real gradient df/dY is linear in g, which can then be called the "scale gradient." Note that the formation of a fractal interface due to the spontaneous appearance of a gradient has already been observed in the corrosion of an aluminum film [13,14].

Of course, the real dynamics of the coasts are more complex than the rapid process considered above. They result from the interplay with the slow weathering processes, generally attributed to carbonation or hydrolysis. These processes act on longer, geological time scales. In order to simulate this evolution, the lithology parameter x_i of all the coast sites is decreased by a small fraction ϵ; i.e., $x_i' = (1 - \epsilon)x_i$ with $\epsilon \ll 1$ after the erosion has stopped at t_f. One or a few coast sites then become weaker than $f(t_f)$, and the rapid erosion dynamics starts again up to the next arrest time. The erosion procedure is then iterated. Snapshots of the coastline at successive arrest times are shown in Fig. 5 together with their measured fractal dimensions. The fluctuations of the measured fractal dimension [obtained in Fig. 5 using a larger value for g than Fig. 4 (left panel)] indicate that the physical model considered here can help to understand observed dimensions of real seacoasts different from 4/3 [15]. It is only for a vanishingly gradient g (in an infinite system) that the universal 4/3 value is recovered.

Moreover, at each restart of erosion, a finite and strongly fluctuating portion of the earth is eroded. The slow weathering mechanisms induce also small fluctuations of $f(t)$ [see Fig. 3 (bottom panel)]. In the language of coastal studies [16], the system state evolves through a dynamical equilibrium where small perturbations (even without special triggering events) may stimulate large fluctuations and avalanche dynamics. This is due to the underlying criticality of percolation systems. In the authors' minds, it is this permanently changing regime which corresponds to the geomorphologic observations of rocky fractal seacoasts.

The above simple model presents limitations, but the simplifications that keep the system in the universality class of percolation are unimportant. The use of different

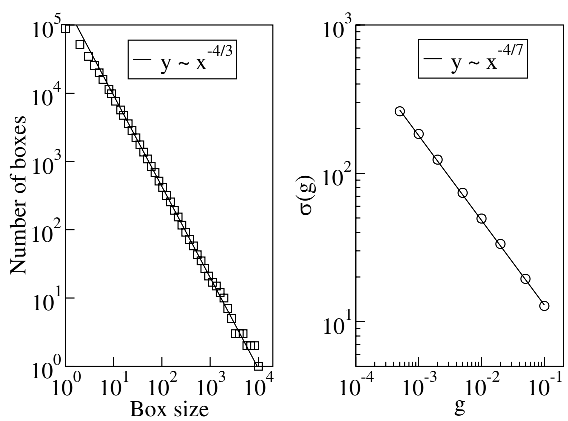

FIG. 4. (Left) box-counting determination of the coast fractal dimension. The straight line is a power law with a slope $-4/3$. The best fit gives $D_f = 1.332(3)$. The data refer to a large system with $L_0 = 10^4$, and a small gradient $g = 10^{-4}$. (Right) scaling behavior of the coast width σ. The straight line is a power law fit with the GP exponent $-4/7$ (each point is an average of over 400 samples with $L_0 = 5000$).

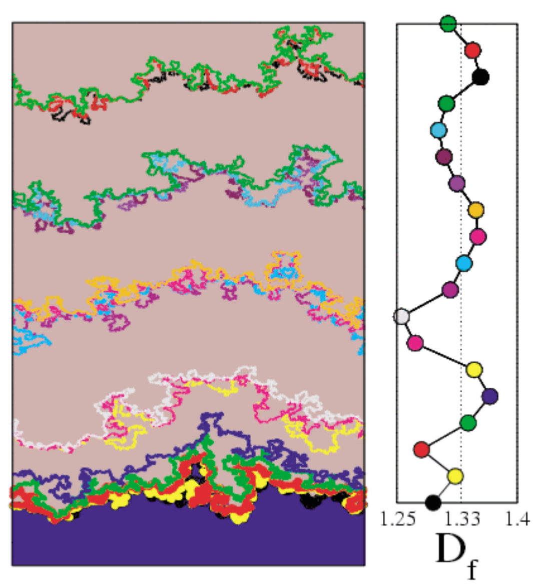

FIG. 5 (color). Snapshots taken during the long term erosion dynamics for a small system ($L_0 = 3000$) with a moderate gradient $g = 0.002$. Note that the measured fractal dimension fluctuates around the universal value 4/3 corresponding to a very small gradient [see Fig. 4 (left panel)]. The color codes for successive arrest times.

weakening rules would eventually modify the dynamics but not the final fractal dimension. The separation between rapid erosion and slow chemical weakening is somewhat arbitrary as both mechanisms can occur simultaneously without changing the coast fractality. Also a better model for damping should take care of a wave frequency dependence as well as it should consider the existence of localization by the frontier of the waves along the irregular coast [7]. This would modify Eq. (2) and change the time evolution. However, since percolation possesses the universality properties of phase transitions [10], the fractal dimension of the coast should not depend on these factors. In the context of corrosion dynamics [14], this has been shown through arguments from dynamical field theory of absorbing states [17]. If large-scale modifications or correlations exist in the lithology properties, then the resulting geometry would be more complex. But our main result, namely, the existence of irregular coasts as a result of a self-stabilization mechanism, would remain correct even though the geometry would be more complex than that of critical percolation. We believe that it is in those terms that the results of several detailed studies of the self-similarity properties of seacoasts can be interpreted [18].

A more radical change in the geometry is expected if sediment transport dominates the shoreline dynamics. In our simplified model, the sediments are supposed to be transported offshore, while in the real erosion process sediments may be partially transported along the coast. There exists, however, many situations in which sediment transport can be neglected. See, for instance, Chap. 18 in [19].

One should also mention that it has been found very recently that the GP power laws apply even when the front is too narrow to be fractal [20]. This extends considerably the range of application of GP to rocky coasts which are irregular, but not to a fractal range.

This work has presented a minimal model for the formation of fractal rocky coast morphology. This model bears on the reciprocal evolution of the erosion power and the topography of the coast submitted to that erosion: the more irregularly eroded the coast is, the weaker the average sea-erosion power. Note that this seems to be an empirically known effect used to build efficient breakwaters that are based on hierarchical accumulations of tetrapods piled over layers of smaller and smaller rocks, in close analogy with fractal geometry [21]. The retroaction leads to the spontaneous formation of a fractal seacoast with a fractal dimension $D_f = 4/3$. The fractal geometry plays here the role of a morphological attractor: whatever its initial shape, a rocky shore will end fractal when submitted to such a type of erosion, forgetting its initial morphology. Note that our model suggests that, on the field, the islands which have resisted to an erosion under a force larger than the final force $f(t_f)$ are stronger that the coast itself. This could be verified on the historical data of known seacoasts and neighboring island evolutions. The model reproduces at least qualitatively some of the features of real coasts using only simple ingredients: the randomness of the lithology and the decrease of the erosion power of the sea. It is worth noting that the use of simple geophysical ingredients leads to an evolution towards a self-organized fractality directly related to percolation theory.

[1] Eric C. F. Bird, *Coasts* (Van Nostrand Rheinhold Co., New York, 1984); *The World Coastline*, edited by Eric C. F. Bird and M. L. Schwartz (Van Nostrand Rheinhold Co., New York, 1985).
[2] E. C. Penning-Roswell, C. H. Green, P. M. Thompson, A. M. Coker, S. M. Tunstall, C. Richards, and D. J. Parker, *The Economics of Coastal Management* (Belhaven Press, London, 1992).
[3] B. B. Mandelbrot, Science **155**, 636 (1967); *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[4] C. C. Barton and C. Dufore, "Fractal Dimension of the Coterminous U.S. Coastline: Scale 1:5,000,000, U.S.G.S. Miscellaneous Investigations Series Map I-XXX," http://coastal.er.usgs.gov/barton/pubs/fractalmap.pdf.
[5] *Coastal Evolution. Late Quaternary Shoreline Morphodynamics*, edited by R. W. G. Carter and C. D. Woodroffe (Cambridge University Press, Cambridge, 1994).
[6] B. Sapoval, *Fractals* (Aditech, Paris, 1989).
[7] B. Sapoval, O. Haeberlé, and S. Russ, J. Acoust. Soc. Am. **102**, 2014 (1997).
[8] B. Hébert, B. Sapoval, and S. Russ, J. Acoust. Soc. Am. **105**, 1567 (1999).
[9] R. A. Davis, Jr., *Oceanography—An Introduction to the Marine Environment*, (William C. Brown Publishing, Dubuque, IA, 1986).
[10] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1991).
[11] T. Grossman and A. Aharony, J. Phys. A **20**, L1193 (1987).
[12] B. Sapoval, M. Rosso, and J. F. Gouyet, J. Phys. (Paris), Lett. **46**, L149 (1985).
[13] L. Balasz, Phys. Rev. E **54**, 1183 (1996).
[14] B. Sapoval, S. B. Santra, and Ph. Barboux, Europhys. Lett. **41**, 297 (1998); A. Gabrielli, A. Baldassarri, and B. Sapoval, Phys. Rev. E **62**, 3103 (2000).
[15] For a recent fractal analysis of the world coastlines, see http://axtnt3.phys.uniroma1.it/Coasts.
[16] S. K. Haslett, *Coastal Systems* (Routledge, Taylor and Francis, London, 2000).
[17] A. Gabrielli, M. A. Munoz, and B. Sapoval, Phys. Rev. E, **64**, 016108 (2001).
[18] M. F. Goodchild, Math. Geol. **12**, 85 (1980); R. Andrle, Earth Surf. Processes Landforms **21**, 955 (1996).
[19] R. A. Davis, Jr. and D. M. Fitzgerald, *Beaches and Coasts* (Blackwell, Oxford, 2004).
[20] A. Desolneux, B. Sapoval, and A. Baldassarri, cond-mat/0302072.
[21] *Shore Protection Manual* (Department of the Army Waterways Experimental Station, Vicksburg, MS, 1984), Vol. 2.