Is the Quantum Hall Effect Influenced by the Gravitational Field?

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Most of the experiments on the quantum Hall effect (QHE) were made at approximately the same height above sea level. A future international comparison will determine whether the gravitational field $\mathbf{g}(x)$ influences the QHE. In the realm of $(1+2)$-dimensional phenomenological macroscopic electrodynamics, the Ohm-Hall law is metric independent ("topological"). This suggests that it does not couple to $\mathbf{g}(x)$. We corroborate this result by a microscopic calculation of the Hall conductance in the presence of a post-Newtonian gravitational field.

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If an experiment is done in Grenoble, France, on the quantum Hall effect (QHE), does it yield the same result as a corresponding experiment done in Boulder, Colorado? Recall that in Grenoble the height above sea level is about 220 m whereas Boulder lies at about 1600 m. For two atomic clocks situated in Grenoble and Boulder, respectively, this difference in heights, and thus the difference in the gravitational potential, yields a measurable effect on the time keeping process [1]. The clock in Grenoble runs slow as compared to the Boulder clock. Accordingly, the influence of the *gravitational* field on effects in *atomic* physics is an established fact, and it seems legitimate to ask whether the QHE is also affected by the gravitational field $\mathbf{g} = \nabla\varphi$, with φ as the potential. The Hall conductance may then depend on the dimensionless quantity φ/c^2.

The QHE is a fascinating manifestation of quantum mechanics on the macroscopic level [2,3]. An important ingredient of the theoretical explanation of the QHE is the idea that the quantized Hall conductance can be linked to a topological invariant, the Chern number [4,5]. The topological interpretation suggests that the quantum Hall resistance (QHR) should be very robust against perturbations. Indeed, its excellent reproducibility makes the QHR very suitable for metrological purposes [6–9]. As pointed out in [10], the reproducibility of the QHR has been established at different locations with a precision of about 5×10^{-8}. However, the locations at which these QHRs were measured are all at about the same height, namely, Gaithersburg, USA (NIST), London (NPL), Sydney (NML)[11]. Therefore, in the future one should compare the results on the QHR as a function of height. This is what we suggest.

Here we advance a macroscopic argument that the QHE and the QHR should be completely independent of the gravitational potential, and we present a microscopic calculation in support of this view.

Conductor in a gravitational field.—The idea that the gravitational field may affect the conductive properties of matter, in particular, those of normal metals or superconductors, is a direct consequence of the fact that charges (here electrons and ions) carry mass and energy. Since gravity is universally coupled to matter, it also acts on the electric currents and the electromagnetic fields in conductors. Because of the equivalence principle, the same qualitative effects should be caused by gravitational as well as by inertial forces. This was analyzed for accelerating (rotating) conductors [12–14], and for the gravitational analog of the Hall effect [15].

Electrodynamics in a four-dimensional spacetime "feels" inertial and gravitational effects via the metric-dependent constitutive (spacetime and material) relations. For conductors, this is manifest in the covariant generalization of Ohm's law [14]. We can use a macroscopic phenomenological picture in order to estimate the possible magnitude of the gravitational effects. For isotropic matter with conductance σ, which is at rest in a Cartesian reference frame, the electric current density 3-vector j^a of the free charges, with $a = 1, 2, 3$, is related to the electric field E_a by means of Ohm's law,

$$j^a = \sigma g^{ab} E_a. \tag{1}$$

Here g^{ab} is the spatial part of the four-dimensional spacetime metric g^{ij}. In the gravitational Schwarzschild field of the Earth with mass M, we have

$$g^{ab} = \left(1 + \frac{GM}{2c^2 r}\right)^{-4} \delta^{ab} \approx (1 - 2\varphi/c^2)\delta^{ab}, \tag{2}$$

with $\varphi = GM/r$. Consequently, when the electric field and the current are measured for a conductor not in a flat space with the Euclidean 3-metric $g^{ab} = \delta^{ab}$, but in a curved spacetime of the Earth, the classical longitudinal conductance is modified by the gravitational field to the effective conductivity $\sigma' = (1 - 2\varphi/c^2)\sigma$. The resulting effect with a relative change of about 10^{-9} is close to the present accuracy of quantum Hall measurements.

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This estimate is based on the macroscopic approach. Another possible manifestation of gravity in conductors arises from the microscopic analysis of the redistribution of charges in conducting matter under the influence of the gravitational or inertial forces. Such a redistribution leads to the weak electric and magnetic fields near the surface of and inside metallic bodies [16,17]. For a conductor in a gravitational field $\mathbf{g}$, the induced electric field is $\mathbf{E} \sim -0.1(m_i/e)\mathbf{g}$, where e is the elementary charge and m_i the mass of the ion in the conductor's lattice.

Hall electrodynamics in ($1+2$*) dimensions.*—Since the 1960s, experimentalists have been able to create a *two-dimensional electron gas* (2DEG) in suitable transistors and, more recently, also in heterostructures at sufficiently low temperatures and to position it in a strong external transversal magnetic field B. Then, the electrons can move only in a plane transverse to B, and one space dimension can be suppressed.

In the following, we formulate a relativistically invariant, effective description of quantum Hall physics that is valid in the sense of a Ginzburg-Landau type theory. In order to accommodate the two-dimensional nature of the electron system, we construct the effective theory in the framework of $(1+2)$-dimensional electrodynamics. We stress that such an approach is not meant to revoke the standard $(1+3)$-dimensional Maxwell-Lorentz theory.

We start from the Maxwell equations valid in any spacetime dimension. In exterior calculus, they can be given compactly as [18] $dG = J$ and $dF = 0$, with G as electromagnetic excitation, J as electric current, and F as electromagnetic field strength. In tensor language [19],

$$\partial_k \mathcal{G}^{ik} = \mathcal{J}^i, \qquad \partial_i F_{k\ell} + \partial_k F_{\ell i} + \partial_\ell F_{ki} = 0, \tag{3}$$

with $\mathcal{G}^{ik} = -\mathcal{G}^{ki}$ and $F_{ik} = -F_{ki}$. This "premetric" form of the Maxwell equations is totally independent of the metric. Thus, all field quantities have different transformation properties: The $\mathcal{J}^i$ is a contravariant vector density [20], $\mathcal{G}^{ik}$ an antisymmetric contravariant tensor density, and F_{ik} a antisymmetric covariant tensor.

In $(1+2)$ spacetime dimensions, the indices i, k, ℓ in (3) run from 0 to 2. Then the current and the field strength can be expressed as

$$(\mathcal{J}^i) = \begin{pmatrix} \rho \\ j^1 \\ j^2 \end{pmatrix}, \qquad (F_{ik}) = \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & B \\ E_2 & -B & 0 \end{pmatrix}, \tag{4}$$

with the area densities of charge ρ and current (j^1, j^2), and electric and magnetic field strengths (E_1, E_2) and B.

We substitute F_{ik} of (4) into (3) and find $\partial_1 E_2 - \partial_2 E_1 + \dot{B} = 0$. The $\mathrm{div} B = 0$ equation is degenerate and drops out. We assumed an infinite extension of the 2DEG. If this is no longer a valid approximation, one has to allow for line currents at the boundary of the sample ("edge currents") in order to fulfill the Maxwell equations.

Being interested in the phenomenology of the QHE, we have to connect in (3) somehow the current $\mathcal{J}^i$ with the field strength F_{ik} by a constitutive law. The linear ansatz of the Ohm-Hall law [18,21,22]

$$\mathcal{J}^i = \sigma^{ik\ell} F_{k\ell} \tag{5}$$

links both quantities in a generally covariant and metric-free form, provided the Hall conductance $\sigma^{ik\ell} = -\sigma^{i\ell k}$ is a contravariant 3-tensor density of rank 3. The totally antisymmetric Levi Civita symbol $\epsilon^{ik\ell} = \pm 1, 0$ has the same transformation property as a tensor density.

If we assume isotropy in three dimensions, we have

$$\mathcal{J}^i = \sigma_{\mathrm{H}} \epsilon^{ik\ell} F_{k\ell}/2, \tag{6}$$

with a scalar field σ_{H}, or, decomposed in time and space,

$$\rho = \sigma_{\mathrm{H}} B, \qquad j^1 = \sigma_{\mathrm{H}} E_2, \qquad j^2 = -\sigma_{\mathrm{H}} E_1. \tag{7}$$

These are classical phenomenological laws that need some interpretation when applied to the description of a quantum system. Within a classical electron model for the conductivity, one finds for the Hall conductivity $\sigma_{\mathrm{H}} = \frac{\rho}{B}$. This relation is expressed in (6) in generally covariant form with a scalar conductivity σ_{H} and with the additional information of a vanishing longitudinal conductivity. However, a vanishing longitudinal conductivity is found in the plateau region of a quantum Hall system, where the Hall conductivity is given by the classical value for a charge density $\rho = \frac{1}{N}\frac{e^2}{h} B$, which corresponds to N completely filled Landau levels. The quantum mechanical input is the robustness of this phenomenology against the influence of disorder and against density variations. With this additional input, the complete independence of the quantum Hall resistance of the gravitational field can be concluded from (6).

We differentiate (6) by ∂_i. Then, because of $\partial_i \mathcal{J}^i = 0$ and $\epsilon^{ik\ell}\partial_i F_{k\ell} = 0$, we find

$$\partial_i \sigma_{\mathrm{H}} = 0; \tag{8}$$

that is, the Hall conductance is constant in time and space.

Equation (6) describes a remarkable constitutive law that is in clear contrast to the standard $\mathcal{G}^{ik} = \sqrt{-g} g^{i\ell} g^{km} F_{\ell m}$ law of $(1+3)$D vacuum electrodynamics. Equation (6) represents a $(1+2)$D Chern-Simons electrodynamics [23], the Lagrangian $\sim \frac{1}{4}\sigma_{\mathrm{H}} \epsilon^{ikl} A_i F_{kl} - A_i \mathcal{J}^i$ of which is *independent* of metric and coframe (here we defined the electromagnetic potential by $F_{k\ell} = \partial_k A_\ell - \partial_\ell A_k$). Thus the energy-momentum tensor of such a model vanishes [22], and the Chern-Simons "fluid", on the right-hand side of Einstein's field equation, cannot act as a source of gravity. By implication, it cannot be affected by the gravitational field either.

Microscopic analysis of the gravitational field dependence of the QHE.— The theoretical analysis presented in this section is concerned with the integer QHE only, whereas the macroscopic discussion in the previous section is applicable to both the integer and the fractional QHE. A magnetic field perpendicular to a two-dimensional electron system leads to a quantization of states in Landau levels (LLs) at energies $E_n = (n + 1/2)\hbar\omega_c$, with $\omega_c = eB/m$. The density of states (DOS) is $\rho(E) = \sum_n \delta(E - E_n)B/\Phi_0$, where the last factor describes the macroscopic degeneracy of an LL. Here, $\Phi_0 = h/e$ denotes the magnetic flux quantum. Thus, a system with a completely filled highest LL is characterized by an excitation gap $\hbar\omega_c$. For the QHR of a system with N completely filled LLs one finds the value $\frac{1}{N}\frac{h}{e^2}$, and as a function of the chemical potential, the Hall resistance follows a stair step curve with plateaus at exactly these values.

Impurities in real samples lead to a broadening of the delta function peaks in the DOS. In addition, electronic states become localized with the exception of a region of delocalized states centered around $E = E_n$. Then, the excitation gap of the clean system is replaced by a mobility gap in the disordered system, which leads to a stair step curve of the Hall resistance as a function of the magnetic field or the electron density. Because of the topological nature of the QHR as an integral over the Brillouin zone, the value of the plateau resistance is unchanged as compared to the clean system [4,5].

Real samples are finite and have contacts. The sample boundaries are described by a confining potential that prevents the electrons from leaving the sample and leads to the formation of one-dimensional edge states [24]. Theoretically, these edge states can be modeled as ideal one-dimensional wires, and assuming the above described localization-delocalization scenario for the bulk states, one can derive the quantized Hall resistance in the framework of the Landauer-Büttiker approach [25]. The current distribution in real experiments is not restricted to narrow channels along the sample edges but generally involves delocalized bulk states as well. For this reason, both the Hall resistivity of bulk states and the resistance of edge states need to be combined for a complete description of experimental results [3].

For reasons of simplicity, we ignore both disorder and edge effects first. We make a quantitative prediction of the influence of gravitational corrections on the Hall resistance of a clean system with a completely filled highest LL. In addition, we argue why the result of this calculation should be valid in the presence of edge states and disorder as well. The influence of gravity up to order g/c^2 is described by the Hamiltonian [26]

$$\hat{H}_{\mathrm{grav}} = -m\varphi(\hat{\mathbf{x}}) - \frac{1}{2m}\hat{\mathbf{p}}\frac{\varphi(\hat{\mathbf{x}})}{c^2}\cdot\hat{\mathbf{p}} - \frac{\hbar}{4mc^2}\hat{\boldsymbol{\sigma}}\cdot(\boldsymbol{\nabla}\varphi(\hat{\mathbf{x}})\times\hat{\mathbf{p}}). \tag{9}$$

We first discuss the situation where $\boldsymbol{\nabla}\varphi(\mathbf{x})$ is perpendicular to the plane of the 2DEG. Labeling the plane of the 2DEG as the xy plane, the gravitational potential depends on z only and hence commutes with $\hat{p}_x$ and $\hat{p}_y$. The first term in $\hat{H}_{\mathrm{grav}}$ turns into a constant that contributes an oscillatory time dependence to the wave functions. The second term, which describes the gravitational redshift of the kinetic energy, does not depend on position any more and modifies only the effective mass of the electrons, which drops out of the calculation of the Hall resistance. The third term is analogous to the Rashba term [27] $\gamma\hat{\sigma}(-i\boldsymbol{\nabla}\times\mathbf{E})$, which describes the influence of the confining electric field on the electron spin. The Rashba term is not known to influence the accuracy of the integer QHE. In addition, the coupling strength $\gamma E \sim 10^{-12}$ eV m is about 23 orders of magnitude larger than the gravitational coupling $\frac{\hbar^2 g}{4mc^2} \sim 10^{-35}$ eV m. Thus, there is no evidence that a gravitational field perpendicular to the plane of the 2DEG influences the Hall resistance to order φ/c^2.

Next, we assume $\mathbf{g}(\mathbf{x})$ to be in the plane of the 2DEG. The electric field needed to counteract the gravitational force is $E = mg/e \sim 10^{-10}$ V/m and hence about 10 orders of magnitude smaller than the typical Hall electric field. The influence of the gravitational field on the ions in a conductor is stronger by a factor of 10^2 and hence more important for actual measurements. The "gravitational Rashba term" in (9) is dominated by the corresponding term due to the externally applied electric field. To discuss the gravitational redshift of the kinetic energy, we assume $\mathbf{g}$ to be oriented along the negative x direction and use a Taylor expansion $\varphi(\mathbf{x}) = \varphi_0 - gx$. We consider a torus with finite extension L_y in the y direction and infinite extension in the x direction.

Using the Landau gauge $\mathbf{A}(\mathbf{x}) = (0, Bx, 0)$, we make the usual product ansatz of a plane wave of momentum k in the y direction and a x dependent $\psi_\alpha(x)$, with $\alpha = (n, k)$ denoting the set of quantum numbers. Upon inserting this ansatz into the full Schrödinger equation, the effective Hamiltonian for $\psi_\alpha(x)$, in the presence of an electric field E in the x direction, reads

$$H_x = \frac{1}{2}\hat{p}_x\left(\frac{1}{m} + \frac{gx}{mc^2}\right)\hat{p}_x + \frac{1}{2}m\omega_c^2(x - X)^2 + eEX + \frac{1}{2}\,\frac{gm\omega_c^2}{c^2}x\left(x - X - \frac{eE}{m\omega_c^2}\right)^2 + \frac{e^2E^2}{2m\omega_c^2}. \tag{10}$$

Here, $X = -\frac{\hbar k}{eB} - \frac{eE}{m\omega_c^2}$. The current in the y direction of a state Ψ_α can be calculated as the derivative

$$I_y = -e\langle\alpha|\hat{v}_y|\alpha\rangle = -(1/eB)\partial\epsilon_\alpha/\partial X. \tag{11}$$

To order $O(g/c^2)$, a perturbative evaluation of the corresponding terms in the Hamiltonian (10) is sufficient. As we want to discuss the influence of disorder and a boundary potential, we first use a transformation that turns the position dependent effective mass [first term in (10)] $1/m(x) = 1/m + gx/(mc^2)$ into an effective potential. Following Gonul *et al.* [28], the position dependent effective mass is replaced by a constant mass m_0 by applying a coordinate transformation $x = f(\tilde{x})$ whose inverse is defined by $\tilde{x} = \int_0^x du\sqrt{m(u)/m_0} = f^{-1}(x)$. This coordinate transformation leaves the spectrum of the Hamiltonian unchanged but renormalizes the wave function and changes the potential $V(x)$ to an effective potential $V_{\text{eff}}(\tilde{x})$. To order $O(g/c^2)$, we find

$$V_{\text{eff}}(\tilde{x}) = V(f(\tilde{x})) + O((g/c^2)^2). \tag{12}$$

An evaluation of the energy shift due to the gravitational corrections yields the current density

$$j_y = \frac{e^2}{h} E_x\left(1 + \frac{g}{c^2}\frac{eE_x}{m\omega_c^2}\right) + \frac{eg}{hc^2}\hbar\omega_c\left(n + \frac{1}{2}\right) \tag{13}$$

for a completely filled nth LL. The constant background current (last term) is, with 10^{-23} A, much smaller than the typical experimental currents of about 10^{-7} A. The nonlinear correction to the Hall conductivity for typical values of the electric field is about 10^{-19}.

The change of the disorder potential due to the transformation (12) is reflected in the change of the correlation function for the disorder potential. However, as the details of the disorder correlator are known to be irrelevant for the derivation of quantum Hall plateaus, we conjecture that the gravitational terms do not change the localization-delocalization scenario responsible for the integer QHE. Similarly, the formation of edge states does not depend on the details of the confining potential, and we find it unlikely that the qualitative properties of edge states are changed by the transformation (12).

In summary, our calculations suggest that the linear Hall resistance is not influenced by a gravitational field to order $O(g/c^2)$. This finding corroborates the macroscopic argument that the Hall resistance may be completely independent of the gravitational field. For a field orientation parallel to the 2DEG, we find both a constant background current and a nonlinear contribution to the Hall current. The only term that is possibly relevant for experiment is the background voltage caused by the gravitational potential. This contribution could be detected with an experimental accuracy of 10^{-8}.

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