Universality of the Shear Viscosity from Supergravity Duals

Alex Buchel

Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J 2W9, Canada and Department of Applied Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada

James T. Liu

Michigan Center for Theoretical Physics, Randall Laboratory of Physics, The University of Michigan, Ann Arbor, Michigan 48109-1120, USA

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Kovtun, Son, and Starinets proposed a bound on the shear viscosity of any fluid in terms of its entropy density. We argue that this bound is always saturated for gauge theories at large 't Hooft coupling, which admit holographically dual supergravity description.

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One of the remarkable connections arising from holography has been the link between black hole thermodynamics and the more traditional case on the field theory side. By working with a black hole (or black brane) background on the gravity side, this allows the investigation of the corresponding gauge theory at finite temperature. Such a connection alone has already yielded many new insights into the thermal phase structure of gauge theories. Of course, it is important to realize that basic equilibrium thermodynamic quantities, such as the free energy and entropy, do not provide complete information about the theory. Nevertheless, with an exact anti–de Sitter/conformal field theory (AdS/CFT) prescription, it ought to be possible to provide dual descriptions of any desired process in the gauge theory.

In practice, one does not expect to find a simple description encompassing all of the information of the gauge theory. However, in keeping with thermodynamic ideas, one would expect that the long-distance fluctuations in the theory will have a hydrodynamic description. In this manner, one may expand the study of gauge theories at finite temperature to include, e.g., transport phenomenon such as diffusion and sound propagation [1–5]. Along these lines, Kovtun, Son, and Starinets (KSS) [6] extended the previous results of [1–3] and investigated the shear viscosity, η, for a large variety of backgrounds.

In the examples of [6], which cover all maximally supersymmetric gauge theories and $\mathcal{N} = 2^*$ gauge theory (to leading order in m/T) [7,8], it was found that the ratio of shear viscosity η to the entropy density s had a fixed value, $\eta/s = 1/4\pi$. On the other hand, weakly coupled systems have $\eta/s \gg 1$, and even common substances have ratios well above this value, which upon reintroducing fundamental constants becomes

$$\frac{1}{4\pi} \rightarrow \frac{\hbar}{4\pi k_B} \approx 6.08 \times 10^{-13}\ \mathrm{K\,s}. \tag{1}$$

Based on these observations, KSS conjectured that there is a universal bound in nature for this ratio, namely [6],

$$\frac{\eta}{s} \geq \frac{1}{4\pi}. \tag{2}$$

It was further argued in [9] that this bound follows from the generalized covariant entropy bound [10].

The intriguing result that $\eta/s = 1/4\pi$ holds exactly for many different nonextremal brane backgrounds suggests that saturation of the bound (2) may always be true for systems admitting a dual supergravity realization. In this Letter, we demonstrate that this is in fact always the case. Before doing so, however, we provide a brief review of the hydrodynamics of strongly coupled systems and the method for extracting η/s from the supergravity dual. We then demonstrate that the bound (2) is saturated for $\mathcal{N} = 2^*$ [Pilch-Warner (PW)] [7], Klebanov-Tseytlin (KT) [11], and Maldacena-Nunez (MN) [12] gauge theory. This finally leads us to the proof that the bound is always saturated in strongly coupled gauge theories admitting a supergravity dual. When the asymptotic supergravity geometry is flat, this result is directly related to the universality of the low-energy absorption cross sections for black holes shown in [13].

Just as in thermodynamics, hydrodynamics is not concerned with the microscopic properties of a theory, but instead in its macroscopic ones. Overall, hydrodynamics may be invoked to provide an effective description of long-wavelength and long time properties of a macroscopic medium. Of particular interest is the study of diffusion governing the flow of, say, heat or charge through a medium. For a charge related to a conserved current, its diffusion is governed by its local concentration, so that $\vec{j} = -D\vec{\nabla} j^0$. Combining this with current conservation, $\partial_t j^0 + \vec{\nabla} \cdot \vec{j} = 0$, then yields the familiar heat equation, $\partial_t j^0 = \vec{\nabla} \cdot (D\vec{\nabla} j^0)$. As expected for a thermodynamic description, these equations are no longer Lorentz invariant, and time reversal invariance is explicitly broken.

While this is well known, its application to AdS/CFT is perhaps less familiar. The important point here is that the diffusion coefficient D is connected to the underlying properties of the gauge theory. At the same time, techniques have been developed to extract D from the fluctuations of long-wavelength modes in the supergravity dual [1–3,6]. So for strongly coupled gauge theories where the dual is known, computation of D and other kinetic coefficients yields additional insight on the nature of the theory itself.

Of present concern is bulk transport through a medium. Here, one works with energy, momentum, and pressure, or, in other words, a conserved stress-energy tensor with components T^{00}, T^{0i}, and T^{ij}. While the analysis is similar to that of charge diffusion, additional complications arise from the tensor nature of $T^{\mu\nu}$. The resulting hydrodynamic quantities of interest include the bulk viscosity ζ, shear viscosity η, and the speed of sound v_s.

In order to compute these kinetic coefficients from the gravity dual, one may in principle extract the appropriate behavior of the boundary stress tensor $T^{\mu\nu}$. Alternatively, as demonstrated in [6], the shear viscosity may be extracted by setting up a "shear perturbation" as a fluctuation on top of the original supergravity background, given by the metric

$$ds^2 = [G_{tt}(r)dt^2 + G_{xx}(r)d\vec{x}^2] + G_{rr}(r)dr^2 + \cdots, \quad (3)$$

where the dual gauge theory has $(t, \vec{x})$ coordinates, r is the transverse coordinate, and the ellipses denote compact directions which are not of direct concern in the following. We take the metric to have a plane-symmetric horizon (extending in p infinite spatial directions, $\vec{x}$) located at $r \to r_0$ where G_{tt} vanishes. The decay of the shear mode is then governed by a diffusion coefficient

$$\mathcal{D} = \frac{\sqrt{-G(r_0)}}{\sqrt{-G_{tt}(r_0)G_{rr}(r_0)}} \int_{r_0}^{\infty} dr \frac{-G_{tt}G_{rr}}{G_{xx}\sqrt{-G}}, \quad (4)$$

denoted the shear mode diffusion constant in [6].

The shear viscosity, η, is obtained from the diffusion constant $\mathcal{D}$ according to [6]

$$\mathcal{D} = \frac{\eta}{\epsilon + P} = \frac{1}{T}\frac{\eta}{s}. \quad (5)$$

Here ϵ, s, P, and T are correspondingly the equilibrium energy and entropy densities, the pressure, and the temperature. As a result, the conjectured shear viscosity bound, (2), is equivalent to the statement

$$\mathcal{D} \geq \frac{1}{4\pi T}. \quad (6)$$

This is the form that will be considered below.

We now compute the shear diffusion constant for a class of backgrounds realizing supergravity duals to 4D gauge theories with eight or fewer supercharges, namely, the PW solution [7], the supergravity dual to the KT cascading gauge theory [11], and the $\mathcal{N} = 1$ MN solution [12]. In all cases, we find that the KSS bound, (2), is saturated.

$\mathcal{N} = 2^*$ gauge theory.—The supergravity dual to $\mathcal{N} = 2^*$, SU(N) gauge theory was proposed in [7] and its nonextremal deformation was studied in [8]. This solution realizes the supergravity dual to $\mathcal{N} = 4$, SU(N) gauge theory softly broken to $\mathcal{N} = 2$. The high temperature thermodynamics of this system thus involves a small parameter $\xi \equiv m/T$, where m is the mass of the $\mathcal{N} = 2$ hypermultiplet, giving rise to the partial supersymmetry breaking. It was observed in [6] that the KSS bound is saturated in this system to leading order in ξ. However, it actually remains saturated for arbitrary ξ.

To see this, we note that the relevant near-extremal 5D Einstein-frame metric involves two functions A and B of a radial coordinate r:

$$ds_5^2 = e^{2A}(e^{2B}dt^2 + d\vec{x}^2) + dr^2. \quad (7)$$

The horizon is taken to be at $r_{\rm hor} = 0$. One of the equations of motion is [Eq. (3.19) of [8]]

$$\ln B' + 4A + B = 4\alpha + \ln\delta, \quad (8)$$

where α and δ are constants specified by the near-horizon asymptotics. This equation may be rewritten as

$$(e^{2B})' = 2\delta e^{4\alpha} e^{-4A+B}, \quad (9)$$

in which case the shear diffusion coefficient may be computed from (4)

$$\begin{aligned}\mathcal{D}_{\rm PW} &= e^{3A}\Big|_{\rm hor}\int_0^{+\infty} dr e^{-4A+B} \\ &= e^{3A}\Big|_{\rm hor}\int_0^{+\infty} d(e^{2B})\frac{1}{2\delta}e^{-4\alpha} = \frac{1}{2\delta}e^{-\alpha}. \quad (10)\end{aligned}$$

Since the black hole temperature is $T = (\delta/2\pi)e^{\alpha}$ [8], we conclude that $\mathcal{D}_{\rm PW} = 1/4\pi T$, which generalizes the result of [6] to all orders in ξ.

KT gauge theory.—The supergravity dual to $\mathcal{N} = 1$ cascading SU($N + M$) $\times$ SU(N) gauge theory was proposed in [11,14], and its nonextremal deformation was studied in [15–17]. Following the notation of [17], the relevant near-extremal 10D Einstein-frame metric involves four functions, x, y, z, and w, of a radial coordinate r and has the form

$$\begin{aligned} ds_{10E}^2 &= e^{2z}(e^{-6x}dt^2 + e^{2x}d\vec{x}^2) + e^{-2z}ds_6^2, \\ ds_6^2 &= e^{10y}dr^2 + e^{2y}(dM_5)^2. \end{aligned} \quad (11)$$

The exact form of $(dM_5)^2$, which depends on the function w but whose volume is independent of w, is unimportant for the present investigation. With this choice of radial coordinate, the horizon is at $r = +\infty$ and the boundary is at $r = 0$.

The system of equations governing the solution leads to the result $x = ar$, where a is a positive constant. The use of (4) as well as this solution for x leads to the shear diffusion coefficient

$$\mathcal{D}_{\mathrm{KT}} = e^{5y+3x-2z}\Big|_{\mathrm{hor}}\int_{\mathrm{hor}}^{\mathrm{bdy}} dr e^{-8ar} = \frac{1}{8a} e^{5y+3x-2z}\Big|_{\mathrm{hor}}. \tag{12}$$

Since the asymptotics of the solution at the horizon, $r \to +\infty$, are $z \to -ar + z_*$ and $y \to -ar + y_*$ [17], we find $\mathcal{D}_{\mathrm{KT}} = (1/8a)e^{5y_*-2z_*}$. Comparing this to the black hole temperature, $T = (2a/\pi)e^{2z_*-5y_*}$ [17], yields the result $\mathcal{D}_{\mathrm{KT}} = 1/4\pi T$.

MN gauge theory.—The supergravity dual to $\mathcal{N} = 1$ SU(N) gauge theory was proposed in [12], and its nonextremal deformation was studied in [18,19]. Here we follow [18], and write the near-extremal 10D Einstein-frame metric as

$$\begin{aligned} ds_{10E}^2 &= c_1(r)^2(\triangle_1^2 dt^2 + d\vec{x}^2) + c_1(r)^2 a^2 ds_6^2, \\ ds_6^2 &= (\triangle_2 r)^{-2}dr^2 + \tfrac{1}{4}h(r)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \tfrac{1}{4}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \tfrac{1}{4}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2. \end{aligned} \tag{13}$$

Using this form of the metric, we compute

$$\mathcal{D}_{\mathrm{MN}} = a^5 h c_1^8\Big|_{\mathrm{hor}}\int_{\mathrm{hor}}^{\mathrm{bdy}} dr \frac{\triangle_1}{\triangle_2 c_1^8 h r} = a^5 h c_1^8\Big|_{\mathrm{hor}}\int_{\mathrm{hor}}^{\mathrm{bdy}} \frac{1}{2Aa^4} d(\triangle_1^2) = a^5 h c_1^8\Big|_{\mathrm{hor}}\frac{1}{2Aa^4}. \tag{14}$$

Note that we have made use of the equation of motion $\triangle_1'\triangle_2 = A/(c_1(r)^8 h(r) r)$ [Eq. (5.55) of [18]], where constant A is the nonextremality parameter. In addition, $\triangle_1$ vanishes at the horizon and becomes unity at the boundary.

From the near-horizon behavior of the metric [Eq. (5.61) of [18]]

$$\begin{aligned} ds_{10E}^2 \approx{}& c_1(r_h)^2(\eta^2 dt^2 + d\vec{x}^2) + \frac{c_1^{18}(r_h)a^2h^2(r_h)}{A^2}d\eta^2 + c_1(r_h)^2a^2\bigg[\frac{1}{4}h(r_h)(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) \\ &+ \frac{1}{4}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{4}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2\bigg], \end{aligned} \tag{15}$$

we find $T = (A/2\pi)(c_1(r_h)^8 a h(r_h)) - 1$. Comparing this with (14), we find as expected $\mathcal{D}_{\mathrm{MN}} = 1/4\pi T$.

Hence we have found the common result that $\mathcal{D} = 1/4\pi T$ for several nontrivial supergravity backgrounds. In all cases, we have taken advantage of reducing the r integral in (4) to a boundary term. This suggests that saturation of the KSS bound is in fact universal, depending only on the thermal nature of the background spacetime.

We now prove the universality of the shear diffusion coefficient. First, observe that, for extremal backgrounds, the Poincaré symmetry of the background geometry ensures that the longitudinal components of the stress tensor can have only the form $T_{\mu\nu} \sim g_{\mu\nu}(\cdots)$. Next note that, while turning on nonextremality involves modifications to the metric as well as to the radial profile of matter fields, this has no effect on the structure of $T_{\mu\nu}$. In particular, $T_t^t - T_x^x = 0$ for both extremal and nonextremal backgrounds. The Einstein equation then gives

$$R_t^t - R_x^x = 0. \tag{16}$$

We now demonstrate that (16) is sufficient to evaluate (4), regardless of the specific background.

Consider a $D = d + p + q$ dimensional background

$$\begin{aligned} ds_D^2 = {}& \Omega_1^2(y)(g_{\mu\nu}(x)dx^\mu dx^\nu) + \Omega_2^2(y)(\hat{g}(z)_{\alpha\beta}dz^\alpha dz^\beta \\ &+ \tilde{g}(y)_{mn}dy^m dy^n), \end{aligned} \tag{17}$$

where $g_{\mu\nu}$ is d dimensional, $\hat{g}_{\alpha\beta}$ is p dimensional, and $\tilde{g}_{mn}$ is q dimensional. An explicit computation yields

$$\begin{aligned} R_{\mu\nu} &= r_{\mu\nu} - g_{\mu\nu}\Omega_1^2\Omega_2^{-2}(\Omega_1^{-1}\tilde{\nabla}^2\Omega_1 + (d-1)\Omega_1^{-2}(\tilde{\nabla}\Omega_1)^2 \\ &\quad + (D-d-2)\Omega_2^{-1}\Omega_1^{-1}\tilde{\nabla}\Omega_1\tilde{\nabla}\Omega_2), \\ R_{\alpha\beta} &= \hat{r}_{\alpha\beta} - \hat{g}_{\alpha\beta}(\Omega_2^{-1}\tilde{\nabla}^2\Omega_2 + (D-d-3)\Omega_2^{-2}(\tilde{\nabla}\Omega_2)^2 \\ &\quad + d\Omega_2^{-1}\Omega_1^{-1}\tilde{\nabla}\Omega_1\tilde{\nabla}\Omega_2), \end{aligned} \tag{18}$$

where $\tilde{\nabla}$ is covariant with respect to $\tilde{g}_{mn}$, and the Ricci tensors $r_{\mu\nu}$ and $\hat{r}_{\alpha\beta}$ are computed from $g_{\mu\nu}$ and $\hat{g}_{\alpha\beta}$, respectively.

We now specialize to nonextremal renormalization group flows of 4D gauge theories. Thus we take $d = 1$, $p = 3$, and $q = 6$ (so that $D = 10$) and furthermore set $r_{\mu\nu} = \hat{r}_{\alpha\beta} = 0$. A nonextremality warp factor $\triangle(r)$ may be introduced by taking $\Omega_1(r) = \Omega_2(r)\triangle(r)$, where these functions depend only on the radial coordinate of $\tilde{g}_{mn}$. For a nonextremal solution, we take the boundary conditions

$$\triangle(r)|_{r=r_0} = 0, \qquad \triangle(r)|_{r=\infty} = 1, \tag{19}$$

where the horizon is at $r = r_0$ and the boundary is at $r = \infty$. In this case, the linear combination of Ricci components, (16), gives rise to $\tilde{\nabla}^2\triangle + 8\Omega_2^{-1}\tilde{\nabla}\Omega_2\tilde{\nabla}\triangle = 0$. Assuming the radial dependence as above, this combination leads to a first integral

$$\frac{d\triangle}{dr}\tilde{g}_{rr}^{-1/2}\tilde{g}_5^{1/2} = A\Omega_2^{-8}, \qquad (20)$$

where A is an integration constant related to the temperature, as will be seen below. Here we have decomposed the 6D metric $\tilde{g}_{mn}$ according to

$$\tilde{g}_{mn}(y)dy^m dy^n = \tilde{g}_{rr}dr^2 + \tilde{g}_{5ij}dy^i dy^j. \qquad (21)$$

It is now easy to see that the expression (4) reduces to

$$\mathcal{D} = \frac{\sqrt{-G(r)}}{\sqrt{-G_{tt}(r)G_{rr}(r)}}\Bigg|_{\text{hor}}\int_{r=r_0}^{r=\infty}\frac{d(\triangle^2)}{2A}$$
$$= \frac{1}{2A}\Omega_2^8(r_0)\tilde{g}_5^{1/2}(r_0). \qquad (22)$$

Furthermore, application of (20) near the horizon yields

$$ds_{10}^2 = \Omega_2^2(r_0)(-\triangle^2 dt^2 + A^{-2}\tilde{g}_5(r_0)\Omega_2^{16}(r_0)d\triangle^2), \qquad (23)$$

from which we can read off the temperature

$$T = \frac{A}{2\pi}\tilde{g}_5(r_0)^{-1/2}\Omega_2^{-8}(r_0). \qquad (24)$$

Combining (22) and (24) finally yields $\mathcal{D} = 1/4\pi T$, thus proving that the KSS bound is always saturated in the supergravity dual, at least to this leading order in corrections.

Of course, as nature has demonstrated, the shear viscosity bound, (2), is not necessarily saturated at weak coupling. In fact, as pointed out in [6], for typical matter (i.e., water under normal conditions) $4\pi\mathcal{D}T \gg 1$. This, however, is not in contradiction with the above proof, as it pertains only to supergravity backgrounds realizing holographic duals to gauge theories at large (strictly speaking infinite) 't Hooft coupling $\lambda \equiv g_{\rm YM}^2 N$.

Although one cannot directly compare strong and weak coupling results, we conjecture that

$$\mathcal{D} = \frac{f(\lambda)}{4\pi T}, \qquad (25)$$

where $f(\lambda)$ is (in principle) a computable function of the 't Hooft coupling, such that for arbitrary λ, $f(\lambda) \geq 1$, and $f(\lambda) \to 1_+$ when $\lambda \to \infty$. Since on the supergravity side of the gauge/string correspondence, 't Hooft coupling corrections translate into string theory α' corrections, verification of this conjecture would involve the study of α' corrections to the hydrodynamics.

Realistically, this can be done for the near-extremal D3-branes. In fact, using the KSS expression, (4), applied to the α'-corrected metric of the near-extremal D3-branes [20], we found that the bound (2) is *violated*:

$$\mathcal{D}T = \frac{1}{4\pi}(1 - 15\gamma + \mathcal{O}(\gamma^2)), \qquad (26)$$

where $\gamma = \frac{1}{8}\zeta(3)(2\lambda)^{-3/2}$. We emphasize, however, that this result assumes that the dispersion relation for the low-energy gravitational shear perturbations which led to the expression (4) is not modified by the α' corrections. This assumption is very likely incorrect, in which case one would have to perform a complete analysis of the metric fluctuations themselves. We hope to return to this issue in the future.

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