

## Exactly Solvable Model of the BCS-BEC Crossover

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We discuss an integrable model of interacting fermions in one dimension that allows a complete description of the crossover from a BCS- to a Bose-like superfluid. This model bridges the Gaudin-Yang model of attractive spin 1/2 fermions to the Lieb-Liniger model of repulsive bosons. Using a geometric resonance in the one-dimensional scattering length, the inverse coupling constant varies from  $-\infty$  to  $+\infty$  while the system evolves from a BCS-like state through a Tonks-Girardeau gas to a weakly interacting Bose gas of dimers. We study the ground state energy, the elementary density and spin excitations, and the correlation functions. An experimental realization with cold atoms of such a one-dimensional BCS-BEC crossover is proposed.

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Starting with the historic controversy between Bardeen and Schafroth about the proper explanation of superconductivity [1], the crossover from a Bardeen-Cooper-Schrieffer (BCS) superfluid with Cooper pairs, whose size is much larger than the interparticle spacing, to a Bose-Einstein Condensate (BEC) of molecules composed of tightly bound fermion pairs has been a basic issue in many-body physics [2]. Very recently, with the observation of molecular condensates near a Feshbach resonance in cold atomic Fermi gases [3], this problem has regained a lot of attention, since for the first time the crossover can be studied in detail. While there is general agreement on the fact that the evolution between the BCS and Bose limit is continuous, all existing theories of the crossover are approximate [4]. In particular, there is no reliable description of the most interesting regime near the Feshbach resonance, where the scattering length diverges. It is therefore of considerable interest to have an analytically soluble model of the BCS-BEC crossover. This is what we will provide in this Letter for the particular case of one dimension (1D).

Cold gases allow us to realize the BCS-BEC crossover by driving a mixture of two spin states with an attractive interaction through a Feshbach resonance, beyond which a bound state appears in the two particle problem in free space. On the BCS side of the crossover, pairs exist only in the many-body system due to the Pauli blocking of states below the Fermi energy. In one and also in two dimensions, the situation is quite different, however, because any purely attractive interaction produces a bound state already at the two particle level. In fact, contrary to the 3D case, its existence is both a necessary and sufficient condition for a BCS instability [5]. As will be shown below, an analog of the 3D crossover can be achieved in 1D by exploiting a confinement-induced resonance (CIR) in a tight trap where the effective 1D scattering length exhibits a resonance caused by the mixing with a closed channel bound state in the trap [6].

We start by considering the Hamiltonian of the Gaudin-Yang (GY) model [7] of a spin 1/2 Fermi gas

interacting via a short range potential  $g_1 \delta(x)$ :

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + g_1 \sum_{i < j} \delta(x_i - x_j), \quad (1)$$

where  $N$  is the total number of fermions and  $m$  their mass. The single dimensionless coupling constant is  $\gamma \equiv mg_1/\hbar^2 n$ , where  $n \equiv N/L$  is the 1D density. For attractive interactions, the Hamiltonian (1) describes a Luther-Emery liquid [8]. When  $\gamma \rightarrow 0^-$ , its ground state is a BCS-like state with Cooper pairs, whose size is much larger than the average interparticle spacing [9]. The strong coupling regime with tightly bound molecules is now simply reached by increasing the magnitude of  $\gamma$ . The resulting fermion pairs behave like a hard core Bose gas, or equivalently like 1D *noninteracting* fermions [10]. In this manner, obviously, one never reaches a weakly interacting BEC as one of the limits of the standard BCS-BEC crossover in 3D. This is a consequence of the trivial fact that the two-body potential  $g_1 \delta(x)$  has a bound state only when  $g_1 < 0$ , but none when  $g_1 > 0$ . As a result, in the regime  $g_1 > 0$ , the ground state of the Hamiltonian (1) is that of repulsive fermions and thus has nothing in common with a weakly interacting Bose gas of molecules.

A qualitatively rather different situation, however, is found for 3D fermions confined in a quasi-1D geometry, where their transverse degrees of freedom are frozen. We assume the fermions to be trapped in a harmonic waveguide with radial frequency  $\omega_{\perp}/2\pi$  and oscillator length  $a_{\perp} \equiv \sqrt{\hbar/m\omega_{\perp}}$ . Having a quasi-1D situation requires that only the lowest transverse mode is occupied. As shown by Bergeman *et al.* [11], the exact solution of the 3D two-body scattering problem in such a waveguide always exhibits one and only one two-body bound state with energy  $\tilde{\epsilon}_b$ , whatever the 3D scattering length  $a$ . Apart from this bound state, all the scattering properties can be described by an effective 1D delta potential  $g_1 \delta(x)$  with strength [6]:

$$g_1 = 2\hbar\omega_{\perp} a(1 - Aa/a_{\perp})^{-1}. \quad (2)$$

As naively expected, an attractive 3D interaction  $a < 0$

implies a negative value of  $g_1$ , associated with a bound state whose energy  $\epsilon_b = -mg_1^2/4\hbar^2$  coincides with the exact bound state energy  $\tilde{\epsilon}_b$  in the limit  $a/a_\perp \rightarrow 0$ . Remarkably,  $g_1$  and thus the binding energy  $\epsilon_b$  remain finite at a Feshbach resonance ( $a = \pm\infty$ ). Entering the positive side  $a > 0$ , however, the vanishing of the denominator at  $a_\perp/a = A \simeq 1.0326$  [12] leads to a CIR, where  $g_1$  jumps from  $-\infty$  to  $+\infty$  just as in a standard 3D Feshbach resonance, as discussed recently also in [13]. Now, for  $g_1 > 0$ , the short range potential  $g_1\delta(x)$  no longer has a bound state, however, it is still present in the quasi-1D problem and its energy  $\tilde{\epsilon}_b$  is always lower than  $-2\hbar\omega_\perp$  [11].

For  $N$  interacting fermions in such a waveguide the quasi-1D condition at zero temperature requires  $\hbar\omega_\perp$  to be much larger than the Fermi energy  $\epsilon_F$ , or equivalently  $(na_\perp)^2 \ll 1$ . Therefore, after the CIR, the true bound state energy  $\tilde{\epsilon}_b$  is the largest energy scale in the problem, implying that the corresponding molecules are unbreakable. The 1D analog of a Feshbach resonance driven BCS-BEC crossover in 3D is therefore described by a modified Gaudin-Yang model (1), where for *positive*  $g_1$  the short range potential is supplemented by an additional bound state with energy  $\epsilon_b = -\infty$ . After crossing the CIR at  $1/\gamma = 0$ , the unbreakable fermion pairs are described by a Lieb-Liniger (LL) model [14] of repulsive bosons. On a formal level, the continuous evolution from an attractive Fermi to a repulsive Bose gas in one dimension is implicit in the Bethe ansatz equations of the GY and LL models. The ground state for both  $\gamma < 0$  and  $\gamma > 0$  may be obtained from the solution of the equations:

$$\pi\rho(k) = 1 + \int_{-Q}^Q \frac{dq}{n} \frac{\gamma\rho(q)}{\gamma^2 + [(k-q)/n]^2},$$

$$\frac{E_0}{N} = \frac{\epsilon_b}{2} + 2 \int_{-Q}^Q \frac{dk}{n} \rho(k) \frac{\hbar^2 k^2}{2m},$$
(3)

where  $E_0$  is the ground state energy and the normalization of  $\rho(k)$ , the quasimomenta distribution,  $\int_{-Q}^Q dk\rho(k) = n/2$  fixes the value of  $Q$ . For  $\gamma < 0$ , they reduce to the GY equations of an attractive Fermi gas while for  $\gamma > 0$ , they reduce to the LL equations of a gas of dimers [15]. This equivalence shows that in the whole regime  $\gamma > 0$ , the density of bosons is  $n_B = n/2$ , their mass is  $m_B = 2m$ , their coupling  $g_B$  is identified with the one of the LL model and  $\gamma \equiv mg_B/\hbar^2 n$ . Since one is now dealing with dimers, the relation between the coupling constant  $g_B$  and the experimentally accessible parameters  $a$  and  $a_\perp$  is no longer given by (2), however. It requires a solution of the dimer-dimer scattering problem in the presence of a transverse confinement. In the limit  $a \ll a_\perp$ , one can use the free space result for the effective scattering length between point like dimers  $a_{3B} \approx 0.6a$  [16], leading to  $g_B \approx 1.2\hbar\omega_\perp a$ . Close to resonance ( $a_\perp/a \gtrsim A$ ), a plausible *ansatz* is

$$g_B = \sqrt{2}\hbar\omega_\perp a(1 - Aa/a_\perp)^{-1}. \quad (4)$$

It follows by assuming that an equation of the form (2) holds also for dimers and the requirement that the resonances match, while  $a_\perp$  is reduced by a factor  $1/\sqrt{2}$ .

The qualitative physics of the modified Gaudin-Yang model is now simple to understand: when  $1/\gamma \rightarrow -\infty$  (BCS limit), the system consists of weakly bound Cooper pairs. The associated excitation gap is related to the spin sector; i.e., there is a finite gap between the singlet ground state and the first triplet excited state. In addition, there are gapless density fluctuations describing the Bogoliubov-Anderson mode of a neutral superfluid. At resonance, when  $1/\gamma = 0$ , the system is a Tonks-Girardeau gas [10] of tightly bound dimers [13]. It still exhibits sound modes with a linear spectrum, however, the spin sector has disappeared because the spin gap diverges. On the positive side of the resonance, the system is an interacting Bose gas of tightly bound molecules. Its excitations are the standard Bogoliubov sound modes, whose velocity vanishes asymptotically in the weak coupling limit  $1/\gamma \rightarrow +\infty$  (BEC limit).

For quantitative results, we start with the ground state where the number of up and down spins is identical. Subtracting the bound state contribution, the relevant finite quantity to consider is  $E_0^{\text{eff}} \equiv E_0 - N\epsilon_b/2$ , which we call the effective ground state energy. This quantity is obtained by numerically solving the Bethe ansatz Eqs. (3) and is plotted as a function of  $1/\gamma$  in Fig. 1. Its asymptotic behavior in the BCS [9] and BEC-limit [14] and near the CIR is

$$\begin{aligned} \frac{E_0^{\text{eff}}}{N\epsilon_F} &\simeq \frac{1}{3} \left[ 1 + \frac{6\gamma}{\pi^2} - \frac{3\gamma^2}{\pi^4} \log^2(|\gamma|) + \dots \right]; \quad 1/\gamma \rightarrow -\infty \\ &\simeq \frac{1}{12} \left[ 1 - \frac{1}{\gamma} + \frac{3}{4\gamma^2} + \dots \right]; \quad 1/\gamma \rightarrow 0 \\ &\simeq \frac{\gamma}{\pi^2} \left[ 1 - \frac{8\sqrt{\gamma}}{3\pi} + \dots \right]; \quad 1/\gamma \rightarrow +\infty, \end{aligned} \quad (5)$$

where  $\epsilon_F \equiv \pi^2\hbar^2 n^2/8m$  is the Fermi energy for the noninteracting gas. We note that on resonance  $E_0^{\text{eff}} = E_0^{\text{eff}}(\gamma = 0^-)/4$ .

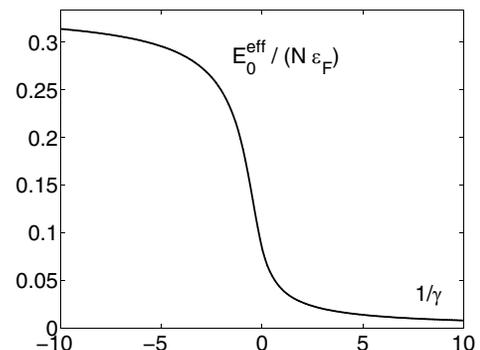


FIG. 1. Ground state effective energy per particle  $E_0^{\text{eff}}/N = E_0/N - \epsilon_b/2$  [in units of the Fermi energy for the noninteracting gas  $\epsilon_F = \pi^2\hbar^2 n^2/8m$ ] as a function of  $1/\gamma$ .

We now discuss the low energy elementary excitations. First consider the density excitations (the “charge sector”) which are gapless phonons with dispersion relation  $\omega = v_c |k|$ , when  $|k| \rightarrow 0$ . An effective theory is provided by a Luttinger liquid description with charge velocity  $v_c$  and correlation functions described by the Luttinger parameter  $K_c$  [17]. When  $\gamma < 0$ ,  $K_c^{(F)}$  describes fermionic correlation functions and is obtained from  $K_c^{(F)} = v_F/v_c$  [18], where  $v_F \equiv \pi\hbar n/2m$  is the Fermi velocity of the noninteracting gas. When  $\gamma > 0$ ,  $K_c^{(B)}$  describes bosonic correlation functions and is given by  $K_c^{(B)} = v_F/2v_c$  [18]. In order to determine these parameters, it is therefore enough to extract the sound velocity from the exact solution. This is done in the usual way by calculating the compressibility from the ground state energy [19]. Its asymptotic behavior in the BCS [9] and BEC-limit [14] and near the CIR is

$$\begin{aligned} \frac{v_c}{v_F} &\simeq 1 + \frac{\gamma}{\pi^2} + \frac{\gamma^2}{2\pi^4} [\log(|\gamma|) - 2] + \dots; 1/\gamma \rightarrow -\infty \\ &\simeq \frac{1}{2} \left[ 1 - \frac{1}{\gamma} + \frac{3}{4\gamma^2} + \frac{3}{4\gamma^3} + \dots \right]; 1/\gamma \rightarrow 0 \\ &\simeq \frac{\sqrt{\gamma}}{\pi} \left[ 1 - \frac{\sqrt{\gamma}}{2\pi} + \dots \right]; 1/\gamma \rightarrow +\infty. \end{aligned} \quad (6)$$

Figure 2 shows  $v_c$  as a function of  $1/\gamma$ . Note that  $K_c$  is always larger than 1, which implies that the system is a 1D superfluid, as discussed, e.g., in [20].

The low energy spin excitations (the “spin sector”) are described by a sine-Gordon model [19] with coupling parameter  $\beta$ . A recent renormalization group analysis of the sine-Gordon model [21,22] shows that when  $\gamma < 0$  the system is driven to the strong coupling fixed point  $\beta^2 = 4\pi$ , i.e., the spin Luttinger parameter  $K_s$  [17] is equal to  $1/2$ . There it becomes equivalent to a noninteracting massive Thirring model [23], i.e., to a gas of massive relativistic fermions. The latter are called massive spinons and may be interpreted as quantum solitons of the sine-Gordon model. They obey the relativistic dispersion relation

$$\omega = \sqrt{(\Delta/2\hbar)^2 + (v_s k)^2}, \quad (7)$$

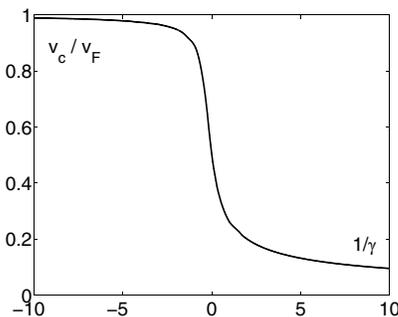


FIG. 2. Sound velocity  $v_c$  [in units of the Fermi velocity for the noninteracting gas  $v_F = \pi\hbar n/2m$ ] as a function of  $1/\gamma$ .

when  $|k| \rightarrow 0$ . Therefore, the low energy part of the spin sector is fully described by the spin velocity  $v_s$  and the spin gap  $\Delta$  or, equivalently, by the mass of the spinon  $m_s \equiv \Delta/(2v_s)$ . The spin gap is also defined as the energy difference between the singlet ground state and the first triplet excited state. The spin velocity and the spin gap can be extracted from the spinon dispersion relation computed from the exact solution of the GY model. The spin gap has the following limiting behavior:

$$\begin{aligned} \frac{\Delta}{\epsilon_F} &\simeq \frac{16}{\pi} \sqrt{\frac{|\gamma|}{\pi}} e^{-\pi^2/2|\gamma|} + \dots; 1/\gamma \rightarrow -\infty \\ &\simeq \frac{2\gamma^2}{\pi^2} \left[ 1 - \frac{\pi^2}{4\gamma^2} + \mathcal{O}(\gamma^{-4}) \right]; 1/\gamma \rightarrow 0^-, \end{aligned} \quad (8)$$

where the BCS limit was already obtained in [9]. When  $\gamma > 0$ , the spin gap is infinite in our model. Its behavior as a function of  $1/\gamma$  is plotted in Fig. 3. When  $1/\gamma < -1$ , it behaves similarly to the gap computed from the BCS mean-field theory  $\Delta_{\text{BCS}} \sim \epsilon_F \exp(-\pi^2/2|\gamma|)$  except for an interaction dependent prefactor  $\sim \sqrt{|\gamma|}$ . The dispersion relation is reminiscent of that of Bogoliubov quasiparticles, provided  $\Delta_{\text{BCS}}$  is identified with  $\Delta/2$  [9]. Near resonance, the spin gap is equal to the modulus of the two-body bound state energy  $|\epsilon_b| = 2\gamma^2 \epsilon_F/\pi^2$ . The smooth crossover from Cooper pairs to molecules occurs when the size of a molecule is of the order of the average distance between particles, which happens for  $\gamma \sim -2$ .

The spin velocity can be computed with the bosonization approach in the weak coupling limit [17] and from the Bethe ansatz equations in the strong coupling limit [9]. It turns out to be given by

$$\begin{aligned} \frac{v_s}{v_F} &\simeq 1 - \frac{\gamma}{\pi^2} + \dots; 1/\gamma \rightarrow -\infty \\ &\simeq -\frac{\gamma}{\pi\sqrt{2}} \left[ 1 - \frac{2}{\gamma} + \dots \right]; 1/\gamma \rightarrow 0^- \end{aligned} \quad (9)$$

and is plotted as a function of  $1/\gamma$  in Fig. 4. Spin correlation functions can be obtained from the knowledge of the spin gap  $\Delta$  and of the charge Luttinger parameters  $K_c$  and  $v_c$ , as shown in [24].

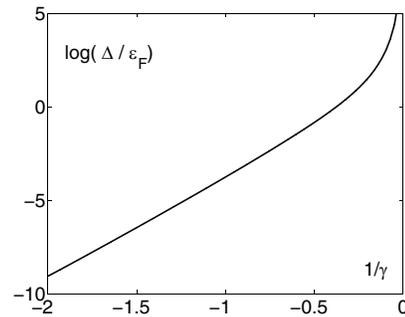


FIG. 3. Spin gap  $\Delta$  [in units of the Fermi energy for the noninteracting gas  $\epsilon_F = \pi^2\hbar^2 n^2/8m$ ] as a function of  $1/\gamma$ .

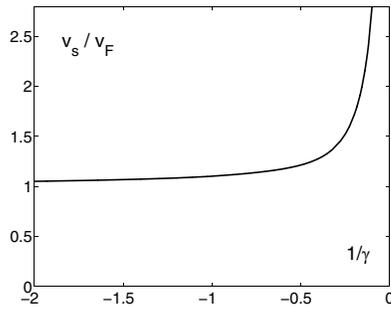


FIG. 4. Spin velocity  $v_s$  [in units of the Fermi velocity for the noninteracting gas  $v_F = \pi\hbar n/2m$ ] as a function of  $1/\gamma$ .

The above scenario for a BCS-BEC crossover can be realized in an experiment with ultracold gases confined in a quasi-1D geometry, e.g.,  $^6\text{Li}$  in an array of 1D tubes created with optical lattices [25] or on an atom chip [26]. In order to reach a 1D regime, the quantum of radial oscillation  $\hbar\omega_{\perp}$  has to be much larger than (i) the quantum of axial oscillation  $\hbar\omega_{\parallel}$ , (ii) the Fermi energy  $\epsilon_F$ , and (iii) the thermal energy  $k_B T$ . Having a degenerate system imposes that the latter is also smaller than the Fermi energy. Note that condition (ii) is required for the validity of the model, as previously mentioned. Taking, e.g., the following values for radial  $\omega_{\perp}/2\pi \sim 100$  kHz and axial  $\omega_{\parallel}/2\pi \sim 200$  Hz trap frequencies, number of particles (per tube)  $N \sim 100$  [25] and temperature  $T \sim 50$  nK [3] allows us to satisfy the previous requirements. To study the crossover in such a setting, the dimensionless coupling constant  $\gamma$  needs to be tuned through a Feshbach resonance [3], giving rise to a CIR as discussed above. The experimental characterization of the different regimes can be done, e.g., by measuring the axial collective modes in the trap. The ratio of the frequencies of the breathing and dipole modes [25] is 2 in both the BCS and CIR limit [13], and  $\sqrt{3}$  in the BEC-limit [27]. In addition, extending an available rf spectroscopy technique used to measure the molecular binding energy [28], it should be possible to extract information on the gap. It is important to note that our model does not describe properly the region between the Feshbach resonance ( $1/\gamma = -\text{Ana}_{\perp}/2$ ) and the CIR ( $1/\gamma = 0$ ). However, due to the condition  $na_{\perp} \ll 1$  of quasione dimensionality this regime can be made arbitrarily small.

In conclusion, our results show explicitly that the evolution from the BCS to the BEC limit is continuous, as expected on general grounds. The system in 1D evolves from a weakly attractive BCS-like Fermi gas to a strongly interacting Tonks-Girardeau gas of dimers and finally to a weakly interacting Bose gas. The model may be realized experimentally by using the combination of a standard Feshbach- with a confinement-induced resonance. In contrast to the 3D case, the exact solution allows to make quantitative predictions in the whole crossover regime.

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