Universal Vortex Formation in Rotating Traps with Bosons and Fermions

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We show that the rotation of trapped quantum mechanical particles with a repulsive interaction can lead to vortex formation, irrespective of whether the particles are bosons or (unpaired) fermions. The exact many-particle wave function constitutes similar structures in both cases, implying that this vortex formation is indeed universal.

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When a system consisting of many interacting particles is set rotating, it may form vortices. This is familiar to us from everyday life: you can observe vortices while stirring your coffee or watching a hurricane. In the world of quantum mechanics, famous examples of vortices are superconducting films [1] and rotating bosonic ^{4}He or fermionic ^{3}He liquids [2,3]. Vortices are also observed in rotating Bose-Einstein condensates in atomic traps [4–6] and are predicted to exist [7] for paired fermionic atoms [8,9]. In this Letter, we show that rotating, trapped particles with a repulsive interaction lead to a similar vortex formation, regardless of whether the particles are bosons or (unpaired) fermions. The exact, quantum mechanical many-particle wave function provides evidence that, in fact, the mechanism of this vortex formation is the same for boson and fermion systems.

Let us now consider a number of identical particles with repulsive interparticle interactions confined in a harmonic trap under rotation. These particles could be electrons in a quantum dot [10], positive or negative ions, or neutral atoms in boson or fermion condensates [11]. Though simple to describe, this quantum mechanical many-body problem is extremely complex and, in general, not solvable exactly. Consequently, in rotating systems the formation of vortices and their mutual interaction is usually described using a mean field approximation. In superconductors this is the Ginzburg-Landau method [1]. For Bose-Einstein condensates, one often applies the Gross-Pitaevskii equation [11,12]. In this way, Butts and Rokhsar [13] found successive transitions between stable patterns of singly quantized vortices, as the angular momentum was increased. A single vortex appears when the angular momentum L is equal to the number of particles N, two vortices appear at $L \sim 1.75N$, and three vortices at $L \sim 2.1N$ (see Refs. [13–16]). For quantum dots in strong magnetic fields, the occurrence of vortices was very recently discussed by Saarikoski *et al.* [17].

Based on the rigorous solution of the many-particle Hamiltonian, we show that striking similarities between the boson and fermion systems exist: the vortex formation is indeed universal for both kinds of particles, and the many-particle configurations generating these vortices are the same. For a small number of particles, the many-body Hamilton operator can be diagonalized numerically. We use a single-particle basis of Gaussian functions to span the Hilbert space. These Gaussians are eigenstates of the trap for radial quantum number $n = 0$ and different single-particle angular momenta. These states dominate for large *total* angular momenta L. We consider only a one spin state, i.e., bosons with zero spin or spin-polarized fermions. Numerical feasibility limits calculations to small particle numbers N. However, the advantages of our approach as compared to mean field methods are that our solutions (i) are exact (up to numeri-

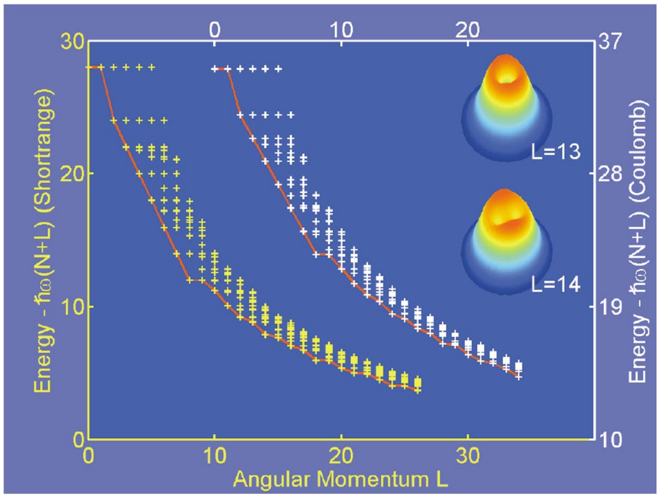

FIG. 1 (color). Rotating bosons in a trap. The figure shows the low-lying many-particle energies of $N = 8$ bosons interacting by a contact interaction (yellow, lower left) or eight charged bosons interacting by the long-range Coulomb repulsion (white, upper right), as a function of angular momentum L (in units of $\hbar$). The red line, also called the "Yrast line," connects the lowest states at fixed L. The insets show vortices in the perturbative densities (as explained in the text), occurring above ratios $L/N = 1$ for a single vortex and $L/N = 1.75$ for two vortices.

cal accuracy), (ii) maintain the circular symmetry and thus have a good angular momentum, and (iii) allow the direct, quantitative comparison between boson and fermion states and thus serve to uncover the origin of the vortices in small systems.

The many-particle energies for rotating clouds of bosons or fermions are compared in Figs. 1 and 2 (bosons and fermions, respectively). The low-lying states are shown as a function of total angular momentum L. Following the tradition in nuclear physics, the line connecting the lowest states at fixed L is called [14] the "yrast" line. (The word yrast originates from Swedish language and means "the most dizzy.") It is marked by a red line in Figs. 1 and 2. For bosons the two spectra appear almost identical, although one of them is calculated with a short-range contact interaction and the other with a long-range Coulomb interaction [18].

The comparison between the lowest energy states of the spectra for bosons and Coulomb-interacting fermions, as displayed in Figs. 1 and 2, reveals striking similarities: The yrast line has the same kinks and vortices can be found in both systems appearing at similar angular momenta, as we explain below.

When studying the appearance of vortices in the boson or fermion densities, we should remember that, in contrast to mean field methods, for an exact calculation with good angular momentum the particle density has circular symmetry and thus does not display the internal structure

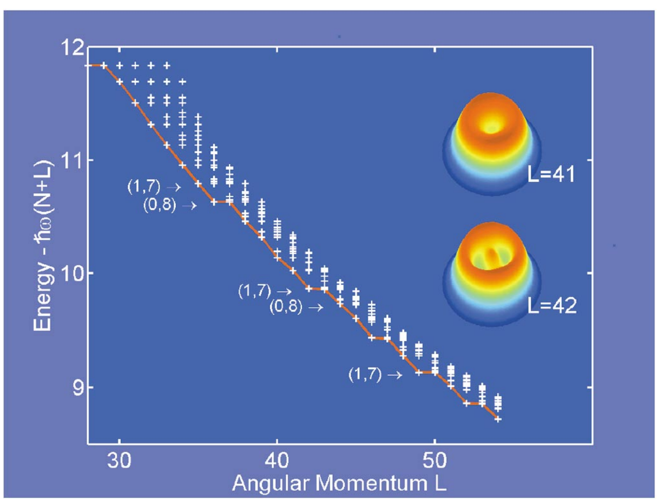

FIG. 2 (color). Rotating, spin-polarized charged fermions (for example, electrons) in a trap. The figure shows the low-lying many-particle energies for $N = 8$. The similarity to the boson case is remarkable. If bosons have a vortex state at angular momentum L_B, the fermions can have the same vortex structure at angular momentum $L_B + N(N-1)/2$. In the example shown here, for fermions a single vortex can be seen in the perturbative densities above $(L-28)/N = 1$. The insets show a single vortex at $L = 41$ and two vortices at $L = 42$, corresponding to $(L-28)/N = 1.75$.

directly. To find the vortices, we therefore would need to study pair-correlation functions. In the fermion case, however, this can be problematic due to the disturbance of the exchange hole. Alternatively, as done in the insets of Figs. 1 and 2, we break the circular symmetry with a small perturbation of the form $V_\ell(r, \varphi) = V_0 \cos(\ell\varphi)$, which has ℓ minima around the center. The perturbation can only couple states which differ in angular momentum by $\pm\ell$. Since the lowest energy state for each angular momentum is also the most important in the perturbation expansion, we can estimate the effect of the perturbation by the mixture of three yrast states $\tilde{\Psi}(L) = \Psi_0(L) + \eta[\Psi_0(L-\ell) + \Psi_0(L+\ell)]$, where L is the angular momentum and η is the mixing parameter, which is of the order 0.1 or smaller. If, for instance, the state in question has a two-vortex structure, then this will appear in $\tilde{\Psi}(L)$ as two distinct minima for $\ell = 2$ already at very small mixing ratio η. The insets of Figs. 1 and 2 are obtained in this way. In the boson case, for $N = 8$ the perturbative densities show a single vortex for $L = 13$, corresponding to $L/N = 1.6$, while a second vortex occurs at $L = 14$, that is $L/N = 1.75$. For fermions, the angular momentum is shifted [19] by $N(N-1)/2$, as explained below. In analogy to the boson case, for $N = 8$ fermions the single vortex still exists at $L = 28 + 13$, while two vortices are found at $L = 28 + 14$.

This universality in the vortex formation can be understood by looking more in detail at the many-particle states of the rotating system. In the case of noninteracting bosons, the many-particle ground state is

$$\Psi_B = e^{-\sum_k |z_k|^2}, \qquad (1)$$

where the coordinates in two dimensions are expressed by complex numbers $z_j = x_j + iy_j$. It turns out that in the case of spin-polarized fermions the corresponding "condensate" is the so-called *maximum density droplet* (MDD) [20]

$$\Psi_F = \prod_{j<k}^{N} (z_j - z_k) e^{-\sum_l |z_l|^2}, \qquad (2)$$

which is a Slater determinant of the consecutive single-particle states, filled from $m = 0$ to $m = N - 1$ and thus has a nonzero angular momentum $L_F = N(N-1)/2$. The state Ψ_F corresponds to the Laughlin wave function for the integer quantum Hall effect [21].

For bosons with short-range repulsive interaction, a single vortex can be formed by multiplying the boson ground state by a symmetric polynomial [15,22] $P_{1V} = \prod_k^N (z_k - z_0)$, where z_0 is the center of mass. If one multiplies the MDD with the same polynomial [23], this gives a good approximation for the exact single vortex state for charged fermions. By noticing that for a system with many particles, the center of mass can be put at the origin, $z_0 = 0$, we can make an ansatz for the state with n fixed vortices forming a ring around the origin:

$$\Psi_{nV} = \prod_{j_1}^{N}(z_{j_1} - ae^{i\alpha_1}) \times \cdots \times \prod_{j_n}^{N}(z_{j_n} - ae^{i\alpha_n})\Psi_{B,F}$$
$$= \prod_{j}^{N}(z_j^n - a^n)\Psi_{B,F}, \tag{3}$$

where $\Psi_{B,F}$ is either the boson condensate or the fermion MDD, and the vortex centers are localized on a ring of radius a ($\alpha_k = k\frac{2\pi}{n}$). This state does not have a good angular momentum, but such a state can be projected out by collecting only terms corresponding to a specific power of the constant a,

$$\Psi_{nV} = a^{n(N-K)} S\left(\prod_{k}^{K} z_k^n\right)\Psi_{B,F}, \tag{4}$$

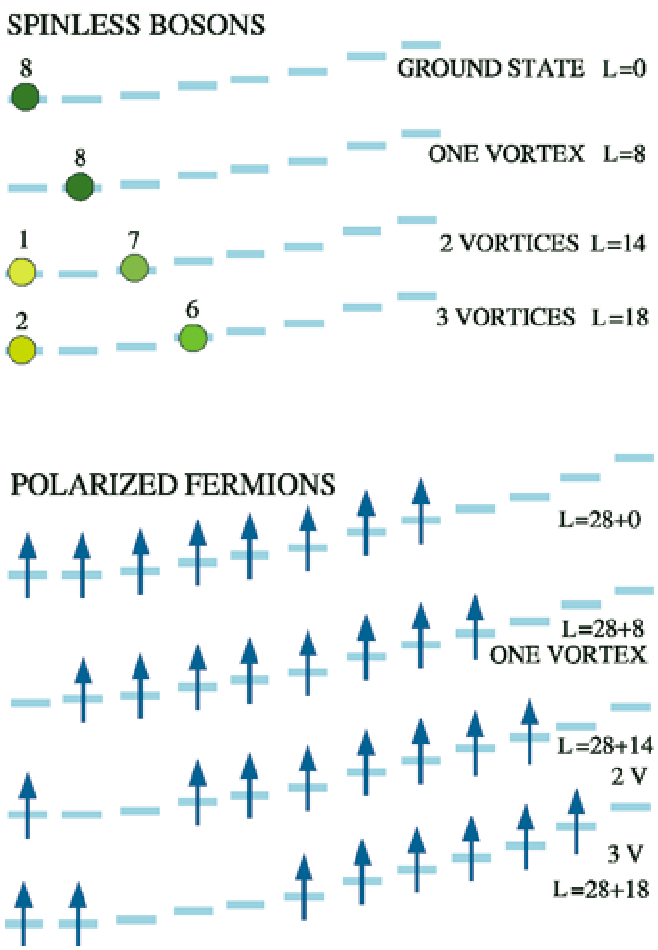

FIG. 3 (color). Vortex generating single-particle configurations for bosons and fermions. (The particle number is chosen to be eight in this example.) Exactly the same excitations of the "ground states" (condensate for bosons, and maximum density droplet for fermions) cause the vortices in both cases, with the same increase in angular momentum.

where S means symmetrization. Note that with $n = 1$, $K = N$, and $a = 0$ this state describes a single vortex fixed at the origin. Figure 3 shows schematically the single-particle occupation of these "vortex generating states" for bosons and fermions. In both cases, the n vortices are generated by exciting K single particles by n units of angular momentum. The quantum states of the numerical exact solutions show that the dominating configurations in cases where we see one, two, or three vortices (independent of the number of particles) are indeed those shown in Fig. 3, for both bosons and fermions.

In the exact diagonalization, by multiplying the exact boson wave function with the wave function Ψ_F of the MDD, Eq. (2), we can determine the overlap between the fermion and boson states. It turns out to be even larger than the weights of the most important configurations. For example, the overlap between the two-vortex states shown in Figs. 1 and 2 is 57%, while the weight of the most important configuration (as shown in Fig. 3) in the boson case is only 15% and in the fermion case 47%.

The vortices are born by the rotational motion and consequently carry angular momentum. In the single-particle picture the angular momentum is associated with the phase of the complex wave function: Going around the angular momentum axis, the phase changes by 2π. Similarly, the phase changes by 2π in going around a vortex center. In the many-particle picture the phase of the wave function depends on the coordinates of all the particles. In this case, the phase change around the vortex cores can be visualized by fixing the coordinates of $N - 1$ particles and plotting the phase as a function of the last coordinate [17]. This is done in Fig. 4 for the vortex generating configuration for $N = 8$. The state Ψ_F has maximum *amplitude*, when the electrons are located so

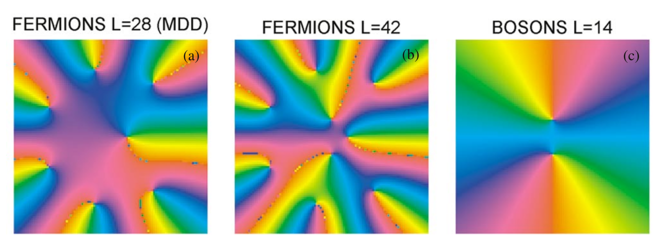

FIG. 4 (color). The phase of the many-particle wave function for eight particles. The phase is shown as a function of the coordinates of one particle with all other coordinates fixed. The color scale is such that the jump from $-\pi$ to π corresponds to a change from orange to pink. (a) shows the phase of the fermion state of the maximum density droplet Ψ_F, Eq. (2), with vortices localized at the fixed electrons; (b) shows the phase of the two-vortex state. The two additional vortices appear in Ψ_{2V}, when the state in (a) is multiplied by the vortex generating polynomial [see Eq. (4)]; (c) shows the same two vortices for the boson state.

that one electron is in the center and seven electrons form a ring around it. To study the phase, we fix six of the electrons on the ring and one at a slightly off-center position in the middle. The resulting phase is shown in Fig. 4(a). One can clearly see that each electron carries a vortex with it, as known from the theory of the integer quantum Hall effect [24]. When the wave function is multiplied with the polynomial generating the vortices, Eq. (4) with $n = 2$, two additional vortices appear [Fig. 4(b)]. When the fermion state Ψ_F is replaced with the corresponding boson state Ψ_B, only the two additional vortices are seen [Fig. 4(c)].

Finally, we will return to the fermion spectrum shown in Fig. 2. The maximum amplitude of the MDD, i.e., the fermion condensate, corresponds to the equilibrium particle positions of a classical system with logarithmic repulsive interactions [21]. In small systems a rigid rotation of this localized state gives some of the high angular momentum states [23,25,26]. For example, in the case of eight particles there are two classical configurations: a single ring of eight particles, which we label by $(0, 8)$, and a ring of seven particles with one particle at the center, $(1, 7)$. The former allows rigid rotation at angular momenta $28 + 8 = 36, 28 + 16 = 44$, etc., while the latter at $28 + 7 = 35, 28 + 14 = 42$, etc., (as marked by arrows in Fig. 2). In the fermion systems, in some cases both localization and vortex structure coincide. This is, for example, the case at $L = 42$ for eight fermions, which in the pair-correlation function shows two vortices as well as very weakly localized particles arranged in a $(1, 7)$ configuration.

To summarize, we have shown with exact solutions of the many-particle systems that the vortex formation in rotating traps of bosons and fermions have universal features: (i) They appear at certain angular momenta determined by the number of particles and number of vortices; (ii) the many-particle excitations generating the vortices are the same for bosons and fermions; and (iii) the vortex formation does not depend on the shape of the repulsive interaction between the particles.

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