Capture into Resonance: A Method for Efficient Control

Dmitri Vainchtein* and Igor Mezić†

Department of Mechanical and Environmental Engineering, University of California, Santa Barbara, California 93106, USA

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To achieve large changes in adiabatic invariants using small control input, a conservative dynamical system must possess an internal resonance. Capture into resonance is an inherently probabilistic process. We propose a control method to make it more structured. We study the motion of charged particles in an electromagnetic field as an example of such a system. When the nominal dynamics brings particles close to a resonance surface, a short control pulse forces the capture of a particle into the resonance with the wave. A captured particle is transported by the wave across the energy levels. The second pulse releases a particle from the resonance when the desired energy level is achieved. We discuss the distribution of energy achieved by the method.

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One of the control objectives for near-integrable Hamiltonian systems is to move phase points from one invariant manifold of the underlying integrable system to another, like changing the energy of a particle (see, e.g., [1–3]). Often, bounded "perturbations" must be used to reduce the cost of control of Hamiltonian systems.

In many near-integrable Hamiltonian dynamical systems major simplifications can be achieved by reducing the coupling between the unperturbed system and weak periodic perturbations to purely resonant interactions occurring in the vicinity of a certain surface in the phase space. A wide range of applications of this technique includes energy exchange between coupled oscillators [4], mixing in fluids [5,6], billiards [7], Josephson junctions [8], and dynamics of charged particles in electromagnetic fields [9]. A theory of the most prominent resonance phenomena, scattering on resonance and capture into resonance, was developed in [10,11].

Consider the dynamics of charged particles in a uniform magnetic field and a weak electrostatic wave. At a qualitative level, this system (and many other systems as well) possesses two regimes of motion: (i) almost free (Larmor) rotation and (ii) captured (resonance) propagation, which are given by two different sets of invariants of motion. Particles spend almost all the time in Larmor rotation—the Larmor radius is related to the adiabatic invariant. Resonance propagation is the only mechanism of significant changes in the adiabatic invariant in conservative weakly perturbed systems, but the transitions between the two regimes are random in nature and thus are rather inefficient as a control mechanism.

We propose a method to structure the transitions with little additional cost. When the internal dynamics brings particles close to a resonant surface a short control pulse forces the capture of selected particles into the resonance with the wave. Captured particles are transported by the wave. The second pulse is applied to release particles from the resonance when the desired energy level is achieved. We show that the proposed mechanism is very sensitive to the accuracy of positioning of the capturing pulse and thus can be used to affect the assigned particles only. The obtained results may be interesting not just for wave-particle interactions, but for a variety of problems where resonant interaction is important and in practical problems such as particle separation.

Consider a charged particle moving in a plasma medium with a uniform magnetic field, $\mathbf{B}$, directed along the z axis in the presence of an electrostatic wave propagating along the y axis, that generates electric field, $\mathbf{E}$:

$$\mathbf{B} = B\mathbf{e}_z, \qquad \mathbf{E} = -E\cos(ky - \omega t)\mathbf{e}_y,$$

where k and ω are the wave vector and the frequency of the wave, respectively, and B and E are constant strengths of the magnetic and electric fields, respectively. The Hamiltonian of a charged particle has the form

$$H = \frac{1}{2m}\left\{P_x^2 + P_z^2 + \left(P_y - \frac{e}{c}Bx\right)^2\right\} + \frac{eE}{k}\sin(ky - \omega t),$$

where $\mathbf{P}$, m, and e are the generalized momentum, the mass, and the charge of a particle, respectively. P_z is an integral of motion and can be set to 0. We introduce a set of nondimensional variables and parameters

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}/\rho, \qquad H_1 = H/mv^2, \qquad \varepsilon = cE/vB, \\ \mathbf{P}_1 &= \mathbf{P}/mv, \qquad t_1 = tvk, \qquad \omega_1 = \omega/kv, \end{aligned} \tag{1}$$

where v is a typical velocity of a particle and $\rho = cmv/(eB)$ is the characteristic Larmor radius. ω_1 is the ratio of the phase speed of the wave to a typical velocity of a particle and in plasma it can be smaller or larger than 1. Here we introduce a phase of the wave φ:

$$\varphi = ky - \omega t, \qquad P_\varphi = P_y.$$

The dimensionless Hamiltonian of a charged particle is (in what follows we do not write the subscript "1")

$$h = \tfrac{1}{2}P_x^2 - \omega P_\varphi + \tfrac{1}{2}(x - P_\varphi)^2 + \kappa\varepsilon\sin\varphi. \tag{2}$$

In (2), $\kappa = 1/\rho$, and

$$h = H/(mv^2) - \omega P_\varphi. \tag{3}$$

The canonically conjugate pairs are now $(x\kappa^{-1}, P_x)$ and (φ, P_φ). We assume that

$$\kappa \ll 1, \qquad \omega, \varepsilon \sim 1.$$

Note that $\kappa\varepsilon$ is twice the ratio of the electrostatic potential energy to the kinetic energy of a particle.

Hamiltonian (2) possesses 2 degrees of freedom. The variable φ is fast and the variables x, P_x, P_φ are slow. In the first approximation we can average the motion over fast φ oscillations. In the averaged system h reduces to

$$h_{\rm av} = P_x^2/2 - \omega P_\varphi + (x - P_\varphi)^2/2.$$

In the averaged system, P_φ and $h_{\rm av}$ are integrals of motion and the problem is integrable. As a characteristic time scale of the averaged system is $1/\kappa$, we introduce slow time $\tau = \kappa t$. The averaging is valid everywhere except for a small part of the phase space where $\dot\varphi \approx 0$. In the (x, P_x, P_φ) phase space the equation $\dot\varphi = 0$ defines a plane parallel to the P_x axis. We call that plane the *resonant surface,* or the *resonance,* and denote it by R:

$$\dot\varphi = \partial h_{\rm av}/\partial P_\varphi = P_\varphi - (\omega + x) = 0. \tag{4}$$

In the next couple of paragraphs we describe the resonance phenomena in uncontrolled systems. We refer the reader to [9–11] for a full description. On R, the projection of the velocity of a particle on the direction of the wave vector is equal to the phase speed of the wave. Near R, φ is not fast compared with x and, therefore, the averaged system does not approximate the exact system adequately. A particle approaches the resonant zone (RZ) (defined as a strip of width $\sim\sqrt{\kappa\varepsilon}$ around R) with the value of P_φ oscillating with a small amplitude $\sim\kappa\varepsilon$ near some value P_φ^-. Upon arrival to RZ, a particle is either captured into resonance or crosses RZ without being captured. After the passage through RZ the value of P_φ oscillates near some other value, P_φ^+, again with a small amplitude $\sim\kappa\varepsilon$ [see Fig. 1(b) below]. Most particles cross RZ in a relatively short time: they are scattered on resonance. The magnitude of the (quasirandom) jump in P_φ, $\Delta P_\varphi = P_\varphi^+ - P_\varphi^-$, is typically of order $\sim\sqrt{\kappa\varepsilon}$. In the case of capture into resonance (CR) upon the arrival to RZ a phase point drifts for a long time (of order $\sim 1/(\kappa\varepsilon)$) along R and P_φ changes by a value of order 1. Whether the particle is captured or not depends sensitively on initial conditions. Hence, it is reasonable to consider CR as a probabilistic phenomenon. The probability of CR in a single passage is small, of order $\sim\sqrt{\kappa\varepsilon}$. In other words, a particle crosses R of the order of $\sim 1/\sqrt{\kappa\varepsilon}$ times before the first CR happens.

Figure 1 shows a single trajectory of the exact system with $\kappa = 0.005$, $\varepsilon = 3$, and $\omega = 1$. Figure 1(a) presents the projection of a characteristic phase curve on the $(x - P_\varphi, P_x)$ plane. The near-circular part is the motion far from the resonance, the dynamics there is governed by the averaged system, and it takes a particle time of order $\sim 2\pi/\kappa$ to complete the period. A wavy near-vertical line is the captured motion. Figure 1(b) illustrates the time evolution of P_φ. Small jumps and a big drop correspond to scatterings and CR, respectively. Note that CR occurs only once in several consecutive crossings of R.

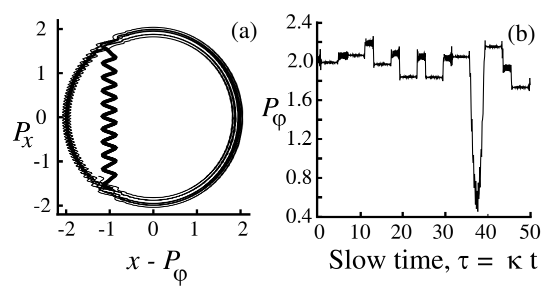

FIG. 1. Scatterings and a capture in the nominal dynamics: (a) projection on the $(x - P_\varphi, P_x)$ plane and (b) P_φ vs slow time.

Near R, introduce $\overline{P}_\phi = P_\phi - (\omega + x)$; see Eq. (4). A phase portrait in the $(\varphi, \overline{P}_\varphi)$ plane can be one of two types: with or without the OD. If $\varepsilon \geq |P_{x,R}|$, where $P_{x,R}$ is the value of P_x on R, phase portraits look like the one in Fig. 2(a). The dashed line Σ is a separatrix. The resonance corresponds to the axis $\overline{P}_\varphi = 0$. If $\varepsilon < |P_{x,R}|$, there is no Σ. Capture is possible only if there is a separatrix on the $(\varphi, \overline{P}_\varphi)$ phase plane. The area under the separatrix loop in the $(\varphi, \overline{P}_\varphi)$ plane, S, is a function of the slow variables, x and P_x, and, hence, changes (although slowly) while a particle moves along a phase trajectory. Suppose the value of S increases. Then if a particle comes very close to Σ, it may cross Σ entering the oscillatory domain (OD) inside the loop. Consequently, instead of leaving the vicinity of the resonance, it starts moving near R along a tight spiral inside the loop [similar to the one shown in Fig. 2(b)]. We say that such particles were captured, and their dynamics is regular:

$$\dot x \approx \kappa P_x, \qquad \dot P_x = \kappa\omega. \tag{5}$$

The value of a resonance action variable, defined as a normalized by 2π area under a phase trajectory in the $(\varphi, \overline{P}_\varphi)$ plane, is conserved during the captured motion:

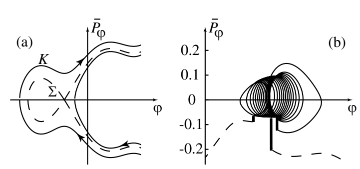

FIG. 2. (a) The schematic phase portrait on the $(\varphi, \overline{P}_\varphi)$ plane for $\varepsilon \geq |P_{x,\rm res}|$; the dashed line Σ is a separatrix. (b) Projection on the (φ, P_φ) plane: forced capture and release.

$$J = \frac{1}{2\pi}\oint \overline{P}_\varphi d\varphi = J_0 = S_0/2\pi = \text{const.} \qquad (6)$$

In Eq. (6), S_0 is the value of S when CR happened.

The fate of a captured particle depends on the behavior of S along a captured trajectory. If in the process of motion $S(t) = S[x(t), P_x(t), P_\varphi(t)]$ returns to S_0, the particle is released from the resonance. In contrast to that, if $S(t)$ does not return to S_0, the captured particle accelerates unboundedly. The values of x, P_x, P_φ at the moment of release as functions of the values at the moment of CR define what is called the "capture input–output function." The structure of capture input–output functions for general Hamiltonians was discussed in [10,11].

Suppose we need to move a particle from some initial state with the "real" energy $H_{1,\mathrm{in}}$ to some final state with the energy $H_{1,f}$. As h is an integral of motion, the objective is equivalent to changing the value of P_φ. In addition, we want to move not all the particles, but only some of them, that satisfy a certain prescribed property. CR is a possible candidate for the job: wait for a while and it will transport some particles to the new energy level. But, as CR is a probabilistic phenomenon one must wait for a long time before it happens and meanwhile the whole dynamics of the particles becomes chaotic. Besides that, captured particles are transported to some new energy level, defined by the input-output function, which might not (and probably will not) be near the target one. Finally, CR chooses a particle at random. Hence, to make CR useful, we need to implement control both at the entry and exit moments to enforce a quick and accurate CR and to secure a timely release.

We use a short pulse in the y direction [recall that $P_\varphi = P_y$; see Fig. 2(b)]. The capturing impulse puts a particle on a certain level of J: $J = J_c$. After CR, a particle oscillates near R. Its dynamics is regular and is governed by Eqs. (5) and (6). If no additional impulses are applied, a particle stays captured as long as $S(t) \geq 2\pi J_c$. When a particle reaches the target value of P_φ we apply another impulse to push the particle from the resonance.

The reachable values of P_φ are given by $\omega^2/2 \leq h + \omega P_\varphi \leq \varepsilon^2/2 + \omega^2/2$, as the trajectories with smaller values of P_φ do not cross R and for larger values of P_φ, trajectories intersect R where there is no separatrix on the $(\varphi, \overline{P}_\varphi)$ phase plane. In principle, one pulse is sufficient to move a particle from any initial value of $P_{\varphi,\mathrm{in}}$ to any final value $P_{\varphi,f} > P_{\varphi,\mathrm{in}}$ within the reachable interval. To achieve this the pulse must put a particle on the level set $J_c = S(P_{\varphi,f})/2\pi$. In this case the "nominal" release happens at $P_\varphi = P_{\varphi,f}$. The larger the value of $P_{\varphi,f}$ the closer the captured trajectory must be to the elliptic fixed point on the $(\varphi, \overline{P}_\varphi)$ phase plane. In the case $P_{\varphi,f} < P_{\varphi,\mathrm{in}}$ there are no requirements on the magnitude of the initial pulse. To release a particle from the capture the second pulse must be applied.

One can apply a sequence of pulses to put a particle on the proper level of J. This control method is robust in a sense that there is no need for extreme accuracy of the magnitude of either capturing or releasing pulses. A capturing pulse must be just strong enough to put a particle inside the OD. If the value of J_c is smaller than $S(P_{\varphi,f})/2\pi$, no adjustments before the releasing pulse are necessary. Otherwise, additional pulse(s) are required to push a trajectory deeper into the OD. A releasing pulse must be just strong enough to kick the particle away from the loop. Characteristic controlled captured dynamics is shown in Fig. 2(b). The right and left dashed lines correspond to nominal motion before and after CR, respectively. The thin solid line is captured motion. Vertical thick solid lines are control impulses. The first pulse (the vertical line in the middle) is the pulse that forced CR. It follows from the shape of the first loop of the captured curve in Fig. 2(b) that it is close to the separatrix. The second pulse (the right vertical line) was applied to put a particle deeper into the OD, and the last pulse (the left vertical line) released a particle from the resonance when a target value of P_φ is reached.

Now we construct a control scheme that does not require feedback, uses kicks localized in a certain place in the real (physical) space, and still does the job for a majority of particles from a cloud of initial conditions. For numerical simulations we chose initial conditions to be $x_0 \in (3.9, 4.1)$, $P_{x,0} \in (-0.1, 0.1)$, $\varphi_0 = 0.5$, and $P_{\varphi,0} = 2.0$. The values of the parameters are $\omega = 1$, $\varepsilon = 12$, and $\kappa = 10^{-4}$. Our objective is to change the value of P_φ from 2.0 to 0.65. The total time of the evolution is $T_f = 5 \times 10^4$. In this case R is a vertical line $x - P_\varphi = -1$. For control kicks we implemented a rectangular pulse that gives an additional term in the equation for $\dot{P}_\varphi$:

$$\dot{P}_\varphi = -\kappa\varepsilon\cos\varphi + p_j f_p(x). \qquad (7)$$

In Eq. (7), the first term comes from the nominal dynamics, the second term is the control pulses with $f_p(x) = 1$ for $|x - x_{k,j}| < \delta$ and $f_p(x) = 0$ otherwise, and $j = \{c, r\}$ correspond to capturing and releasing pulses, respectively. We call x_k, δ, and p location, width, and magnitude of the kick, respectively. We used $\delta = 10^{-3}$, $p_c = -0.003$, $x_{k,r} = -0.4$, and $p_r = 0.001$ and varied $x_{k,c}$ as discussed below. A typical captured trajectory is shown in Fig. 3. The upper and lower horizontal lines in Fig. 3(b) denote locations of the capture and the release, respectively. Comparing Fig. 3 with Fig. 1, note that the difference in characteristic values of the slow time reflect that CR happened in the first period. The energy applied by the kicks is approximately 0.015, 2 orders of magnitude less than the difference between $P_{\phi,f}$ and $P_{\phi,\mathrm{in}}$.

Our simulations show that the percentage of captured particles is rather sensitive to $x_{k,c}$ and is less sensitive to δ_c and p_c. The best moment to apply the kick is when a particle is right above the separatrix loop [point K in

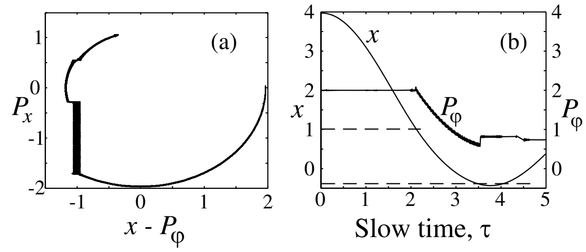

FIG. 3. Forced capture and release. (a) Projection of the phase curve on the $(x - P_\varphi, P_\varphi)$ plane; (b) the evolution of x and P_φ. The upper and lower horizontal dashed lines denote the locations of the capture and the release, respectively.

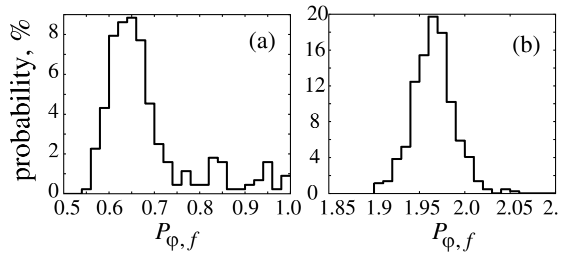

FIG. 4. Histograms of the distribution of the final values of P_φ for $x_{k,c} = 1.02$. (a) $m = m_1$; (b) $m = m_2$.

Fig. 2(a)]. For different initial conditions particles arrive to the respective points K at different values of x. Hence, the kick is most effective (in terms of the percentage) when it is applied in the middle of a range of x_K for a given set of initial conditions. In terms of the values of the parameters used in simulations, the best results were achieved with $x_{k,c} \approx 1.02$, for which 87% of trajectories were captured. For $x_{k,c} = 1.01$ and $x_{k,c} = 1.03$, the percentage of captured trajectories was 53% and 38%, respectively.

The mechanism of release is more robust—if the pulse is large enough the release definitely occurs. All the captured trajectories (for which the releasing kick was applied) were released. After the release particles proceed along an averaged trajectory. Before the time of simulations runs out, they undergo an additional scattering on resonance. In that region the area under the separatrix decreases and the natural CR is impossible.

The sensitivity of the probability of CR to the match between the locations of the pulse and the resonance suggest possible applications of the control via CR. One of the possible applications is to separate particles of two different types that differ by mass only: m_1 and m_2. We performed a set of simulations with $\beta = m_2/m_1 = 1.05$. For the type "2" particles the resonance is located at $x = 2 - 1/\beta$. As a result, the pulse, which is synchronized with the type 1, is applied at a "wrong" moment. Consequently, only a few (of the order of 3%) of the type 2 particles are captured due not to a control pulse, but a natural dynamics. Histograms of the final distributions of the values of P_φ are shown in Fig. 4. The distribution in Fig. 4(a) clearly has a peak near the target value of 0.65. The distribution in Fig. 4(b) has a peak value of 1.97 near the original value of 2.0. The width of both distributions is of order $\sim\sqrt{\kappa\varepsilon}$. Both distributions can be made sharper by reducing κ and/or ε. Similarly, particles can be separated based on the initial discrepancy in energy or coordinates.

In conclusion, we proposed a method to control transitions between different regimes of Hamiltonian dynamics (in particular, resonance transitions between Larmor and captured motions of charged particles in electromagnetic fields) by applying weak control pulses. The sensitivity of the proposed mechanism to initial conditions allow for an accurate addressing of the control.

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*Electronic address: dmitri@engineering.ucsb.edu
†Electronic address: mezic@engineering.ucsb.edu

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