Hall Effect of Light

Masaru Onoda,[1,*] Shuichi Murakami,[2,†] and Naoto Nagaosa[1,2,‡]

[1]*Correlated Electron Research Center (CERC), National Institute of Advanced Industrial Science and Technology (AIST),
Tsukuba Central 4, Tsukuba 305-8562, Japan*
[2]*CREST, Department of Applied Physics, University of Tokyo, Bunkyo-ku, Tokyo 113-8656, Japan*

We derive the semiclassical equation of motion for the wave packet of light taking into account the Berry curvature in momentum-space. This equation naturally describes the interplay between orbital and spin angular momenta, i.e., the conservation of the total angular momentum of light. This leads to the shift of wave-packet motion perpendicular to the gradient of the dielectric constant, i.e., the polarization-dependent Hall effect of light. An enhancement of this effect in photonic crystals is also proposed.

The similarity between geometrical optics and particle dynamics has been the guiding principle to develop the quantum mechanics in its early stage. One can consider a trajectory or ray of light at the length-scale much larger than the wavelength λ. By setting $\hbar = c = 1$, the equation for the eikonal ψ in the geometrical optics reads $[\partial_\mu \psi(x)]^2 = [\nabla \phi(r)]^2 - n(r)^2 = 0$, where $x_\mu = (r, t)$ is the four-dimensional coordinates, $\psi(x) = \phi(r) - n(r)t$, and $n(r)$ is the refractive index slowly varying within the wavelength [1]. This equation is identical to the Hamilton-Jacobi equation for the particle motion by replacing ψ by the action S. The equation for the ray of light can be derived from the eikonal equation in parallel to the Hamiltonian equation of motion. Deviation from this geometrical optics is usually treated in terms of the diffraction theory. However, most of the analyses have been done in terms of the scalar diffraction theory, neglecting its vector nature. However, the light has the degree of freedom of the polarization, which is represented by the spin $S = 1$ parallel or antiparallel to the wave vector k. It is known that this spin produces the Berry-phase when the light is guided by the optical fiber with torsion [2]. Also the adiabatic change of the polarization even without the change of k produces the phase change called the Pancharatnam phase [3]. In this Letter, we will show that the trajectory or ray of light itself is affected by the Berry phase [4], which leads to various nontrivial effects including the Hall effect.

For simplicity, we focus on an isotropic, nonmagnetic medium; the refractive index $n(\mathbf{r})$ is real and scalar, but is not spatially uniform. Let us consider the wave function for the wave packet with the wave vector centered at k_c and the position centered at r_c. Because of the conjugate relation of the position and wave vector, both of them inevitably have a finite width of distribution. Therefore, the relative phase of the wave functions at k and $k + dk$ matters, which is the Berry connection or gauge field Λ_k defined by $[\Lambda_k]_{\lambda\lambda'} = -ie_{\lambda k}^\dagger \nabla_k e_{\lambda' k}$ where $e_{\lambda k}$ is the polar-

ization vector of λ-polarized photon, and $\lambda = \pm$ correspond to the right- and left-circular polarization. Here, it is noted that the SU(2) gauge field Λ_k is 2×2 matrix. The corresponding Berry curvature (BC) or field strength is given by $\Omega_k = \nabla_k \times \Lambda_k + i\Lambda_k \times \Lambda_k$. Because of the masslessness of a photon, it is diagonal in the basis of the right and left-circular polarization as given by $\Omega_k = \sigma_3 \mathbf{k}/k^3$ (a massive case will be discussed later). It corresponds to the field radiated from the monopole with strength ± 1 located at $\mathbf{k} = 0$. We pick up a correction to the geometric optics up to the linear order in $|\nabla n(r_c)|/k_c$. An equation of motion (EOM) is derived including an effect of Λ_k in the similar way as the Bloch waves of electrons [5]. By considering the effective Lagrangian for the center coordinates r_c and k_c, the variational principle leads to the following EOM [5,6]:

$$\dot{\mathbf{r}}_c = v(\mathbf{r}_c)\frac{\mathbf{k}_c}{k_c} + \dot{\mathbf{k}}_c \times (z_c|\Omega_{\mathbf{k}_c}|z_c), \tag{1}$$

$$\dot{\mathbf{k}}_c = -[\nabla v(\mathbf{r}_c)]k_c, \tag{2}$$

$$|\dot{z}_c) = -i\dot{\mathbf{k}}_c \cdot \Lambda_{\mathbf{k}_c}|z_c), \tag{3}$$

where $v(\mathbf{r}) = 1/n(\mathbf{r})$ is the velocity of light, and $|z) = {}^t[z_+, z_-]$ represents the polarization state. This is the central result of this Letter, from which the Hall effect of light is derived below in a straightforward way.

The limit of geometrical optics is obtained by neglecting the terms containing Λ_k or Ω_k, reproducing Fermat's principle. Let us study phenomena caused by Λ_k and Ω_k. First, the Λ_k term in (4) causes the phase shift by the directional change of the propagation discussed in ref. [2], as seen in the following way. The equation for $|\dot{z}_c)$ gives the solution $|z_c^{\text{out}}) = {}^t[e^{-i\Theta}z_+^{\text{in}}, e^{i\Theta}z_-^{\text{in}}]$, where $|z_c^{\text{in}}) = {}^t[z_+^{\text{in}}, z_-^{\text{in}}]$ is the initial state of polarization. Θ is the solid angle made by the trajectory of momentum: $\Theta = \oint d\mathbf{k} \cdot [\Lambda_k]_{++} = \int_S d\mathbf{S}_k \cdot [\Omega_k]_{++}$ where $d\mathbf{S}_k$ is the surface element in k space and S is a surface surrounded by the

trajectory. This is the phase shift by the directional change of propagation [2].

On the other hand, the second term in Eq. (1) induces a new effect found in this Letter; the trajectory of light is affected by $\mathbf{\Omega}_k$. We note that this term guarantees the conservation of total angular momentum of photon $j_z = [\mathbf{r}_c \times \mathbf{k}_c + (z_c|\sigma_3|z_c)(\mathbf{k}_c/k_c)]_z$, assuming the rotational symmetry around the z axis. From Eqs. (2)–(4), one can prove $dj_z/dt = 0$ when $\epsilon(\mathbf{r}) = \epsilon(\sqrt{x^2 + y^2}, z)$. Note that the orbital angular momentum $\mathbf{r}_c \times \mathbf{k}_c$ is defined only when the position $\mathbf{r}_c$ is well defined, which necessarily leads to the finite distribution of the wave vector and hence to the Berry phase. This argument applies to a light beam, whereas it does not for the plane wave state.

This Berry-phase term induces the Hall effect of light. Let us first consider reflection and refraction of light at a flat interface between two media with different dielectric constants. Let $z = 0$ be the boundary between the two dielectric media with $n = v_0/v_1$ where v_0 and v_1 are the velocities of light in the media below and above the boundary, and $\mathbf{r}_c$ and $\mathbf{k}_c$ of the incident light are within the $y = 0$ plane (Fig. 1).

The Snell's law tells us $\sin\theta_T/v_1 = \sin\theta_I/v_0$ for the transmitted light and $\theta_R = \pi - \theta_I$ for the reflected light, where $\theta_{I,T,R}$ are the angle between the z axis and $\mathbf{k}_c$ of the incident, transmitted and reflected lights. This relation can be derived from Fermat's principle in geometrical optics. The question is the following: What is the deviation from this Snell's law due to the Berry-phase? The answer is the transverse shifts of the centers of the reflected and the transmitted wave packets. This phenomenon could be regarded as the ''Hall effect of light,'' wherein the ''force'' to the light, i.e., the change of the dielectric constant, is perpendicular to the shift. The shift of the transmitted light can be easily calculated from th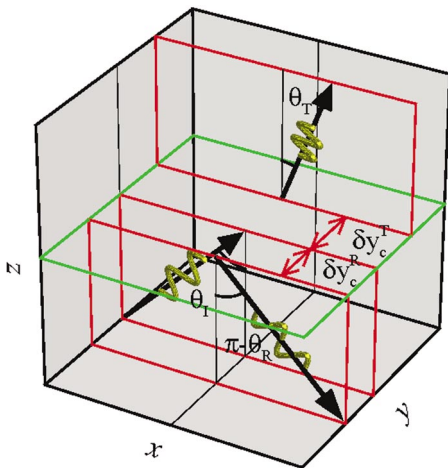e EOM. Strictly speaking, the EOM can be applied only when the refractive index n varies much slower than the wavelength; thus the EOM is not reliable when the interface is sharp. Nonetheless, it turns out to give the correct answer for the shift of the transmitted light even for the sharp interface, because the shift is governed by the conservation law of the z component j_z of total angular momentum. We assume that j_z is conserved in reflection and in refraction separately: $j_z^I = j_z^T$, $j_z^I = j_z^R$, where the superscripts I, T, and R stand for incident, transmitted, and reflected lights, respectively. This is required when the light is regarded as a quantized object, photon; each photon is reflected or refracted with some probabilities. From this conservation of j_z, the transverse shifts of transmitted and reflected wave packets are estimated as

$$\delta y_c^A(\theta_I) = \frac{[(z_c^A|\sigma_3|z_c^A)\cos\theta_A - (z_c^I|\sigma_3|z_c^I)\cos\theta_I]}{k_c^I \sin\theta_I}, \quad (4)$$

where $A = T$ or R, k_c^I is the momentum of the incident light, $|z_c^{I,T,R})$ stand for the polarizations of the incident, transmitted and reflected photons. We note that $|z_c^{T(R)})$ is obtained from $|z_c^I)$ and the transmittances (reflectances) of two eigenmodes with linear polarizations.

It has been already known that the totally reflected light beam undergoes shifts of position in the plane of incidence (longitudinal) or normal to the plane of incidence (transverse). The longitudinal shift was first studied by Goos and Hänchen [7]. The transverse shift was predicted by Fedorov [8], and have been observed experimentally by Imbert [9], followed by a number of theoretical approaches [10,11]. The transverse shift occurs not only in total reflection, but also in partial reflection and refraction [12]. While there are several papers on theoretical predictions [12–14], they contain rather complicated calculations and the issue remains still controversial. The shift of the transmitted light has not been

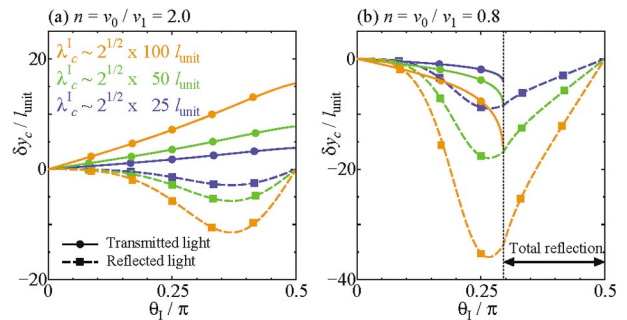FIG. 1 (color). Transverse shift of light beams in the refraction and reflection at an interface.

FIG. 2 (color). Transverse shift in the center of the transmitted and reflected wave packets of light for (a) $n = 2.0$ and (b) $n = 0.8$ with the incident wave packet being right-circularly polarized. The solid and dashed lines for the analytic results Eq. (4). The solid circles and squares are the results of simulation. $\lambda_c^I = 2\pi/k_c^I$ is the wavelength of incident wave packet. l_{unit} is an arbitrary length scale. The sign of the shifts is reversed for the left-circularly polarized light.

experimentally measured to the authors' knowledge. Our derivation based on the EOM is much clearer and simpler. Also the result is supported by the numerics described below, showing a wide applicability of our EOM.

In order to test our prediction Eq. (4), we have numerically solved the Maxwell equation. It is found that both the reflected and transmitted wave packets are shifted along the y or $-y$ direction depending on the circular polarization, which perfectly agrees with Eq. (4) as shown in Fig. 2. This phenomenon is analogous to the spin Hall effect [15,16], where the Berry-phase from band structure gives rise to spin-dependent motion of electrons.

It is noted that the similar effect is expected also for massive particle waves with intrinsic angular momentum. However, the effect is suppressed by the mass factor, m. For example, for the spin-1/2 massive field, the BC in helicity basis is given by $\mathbf{\Omega}_k = \frac{1}{2E_k^2}[\frac{m}{E_k}(\sigma_1 e_\theta + \sigma_2 e_\phi) + \sigma_3 e_k]$, where $E_k = \sqrt{k^2 + m^2}$ and $e_{k,\theta,\phi}$ are the unit vectors of the spherical coordinate system in $\mathbf{k}$ space.

The BC of light is proportional to $1/k^2$. Consequently, the transverse shift is a fraction of the wavelength, and is small for the visible light. In the following we propose a method to enhance the Hall effect of light by use of photonic crystals [17]. As is known, the periodic modulation of the dielectric constant leads to formation of Bloch waves of light analogous to those of electron in solids. The degeneracy of right- and left-circularly polarized light is generally lifted. Hence, the SU(2) gauge invariance breaks down to U(1) (Abelian). This is similar to the anomalous Hall effect in ferromagnetic materials [18]. In photonic crystals, monopoles in $\mathbf{k}$-space can exist away from $\mathbf{k} = \mathbf{0}$. The anomalous velocity is enhanced when $\mathbf{k}$ is near the monopole.

For an explicit example, we consider a two-dimensional (2D) photonic crystal with the dielectric function $\epsilon(\mathbf{r})$, $\frac{1}{\epsilon(\mathbf{r})} = \{4/[3(5 + 12|\xi| + 8\xi^2)]\} \sum_{i=1}^{3} \{[\xi - \cos(\mathbf{b}_i \cdot \mathbf{r} + \frac{2\pi}{3})]^2 + [\xi + \cos(\mathbf{b}_i \cdot \mathbf{r} - \frac{2\pi}{3})]^2\}$, where $\mathbf{b}_1 = (\frac{2\pi\sqrt{3}}{3a}, -\frac{2\pi}{3a})$, $\mathbf{b}_2 = (0, \frac{4\pi}{3a})$, and $\mathbf{b}_3 = -\mathbf{b}_1 - \mathbf{b}_2$. The spatial distribution of $\epsilon(\mathbf{r})$ is shown in Fig. 3(a). For sim-
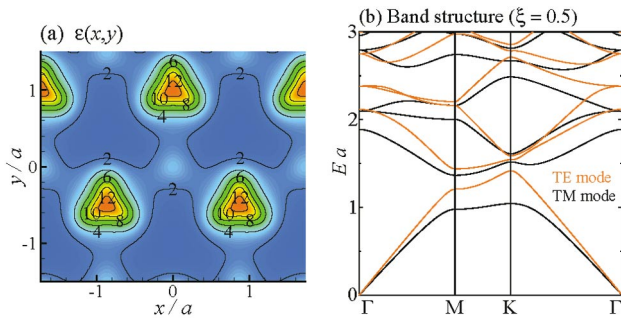
plicity, we shall focus on the case with $k_z = 0$. In other words, we shall consider wave packets extended in z direction, i.e., wave ribbons. The eigenmodes are classified into transverse magnetic (TM) modes and transverse electric (TE) modes. For the TM modes, the magnetic field is in the xy-plane, while for the TE modes it is in the z direction. The photonic bands are shown in Fig. 3(b). The BC is along the z direction in 2D, and its z component is shown in Fig. 4. It is seen that $\mathbf{\Omega}_k$ is strongly enhanced near the corners of the Brillouin zone, which can be interpreted as a 2D cut of the monopole structure in the extended space including the parameters, e.g., ξ in the present case (ξ represents the degree of inversion-symmetry breaking).

To induce the Hall effect of light in photonic crystals, we have to introduce a gradient of the refractive index, as seen in Eq. (2). Therefore, we shall introduce a slow variation of the envelope of the refractive index as $n(\mathbf{r}) \rightarrow n(\mathbf{r})/\gamma(x)$ where $1/\gamma(x)$ changes from 1.0 to 1.2 within the range of $w = 10a$. We have solved the EOM for the wave packets in the photonic crystal, $\dot{\mathbf{r}}_c = \gamma(x_c)\nabla_{\mathbf{k}_c} E_{n\mathbf{k}_c} + \dot{\mathbf{k}}_c \times \mathbf{\Omega}_{n\mathbf{k}_c}$, $\dot{\mathbf{k}}_c = -[\nabla\gamma(x_c)]E_{n\mathbf{k}_c}$, where E_{nk} and $\mathbf{\Omega}_{nk}$ are the energy and the BC of the nth band in the unperturbed crystal. The EOM for $|z\rangle$ brings about the phase shift for each mode. We set the initial $\mathbf{k}_c$ near the corners of the Brillouin zone. This is because the BC is large at the corners and the effect by the anomalous velocity is expected to be prominent. The obtained trajectories are shown in Fig. 5. It is found that the shift of $\mathbf{r}_c$ reaches to dozens of times the lattice constant especially for the TE second band.

We note that the BC is generally present in photonic crystals if the inversion-symmetry is broken. Our EOM offers an easy method to calculate the trajectory of wave packets for generic photonic crystals [19]. By contriving crystal structures, this effect can be enhanced consider-

FIG. 3 (color). (a) Dielectric function and (b) band structure of the 2D photonic crystal. The Brillouin zone is shown in Figs. 4 and 5(b).
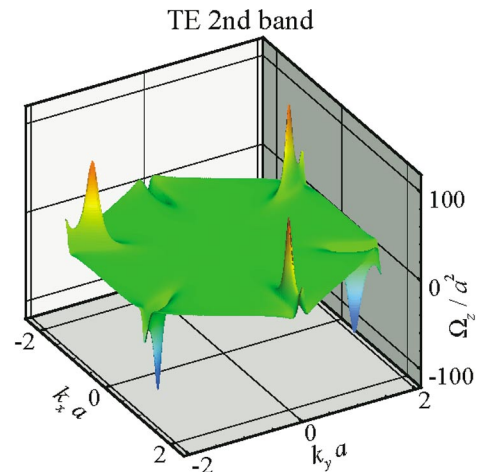

FIG. 4 (color). Example of the BC of the 2D photonic crystal. It is prominent at the corners of the Brillouin zone.
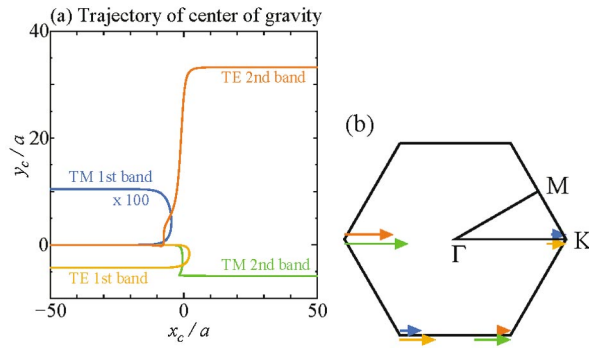
FIG. 5 (color). (a) Real-space trajectory of the center of the wave packet. (b) Momentum-space trajectory of the center of the wave packet. The arrows in the same colors in (a) and (b) refer to the same photonic modes.

ably as follows. The BC around the zone corners is determined mostly by the splitting, $2|\Delta|$, between the neighboring bands. Suppose, for example, at $k = k_c$ another band comes very close in energy to the one considered. The BC is approximately given by $\Omega_z \sim [v^2\Delta/(\Delta^2 + v^2|k - k_0|^2)^{3/2}]$, where v is a nominal velocity of light. Thus when the light traverses this k_0 point, the shift is given by $\delta y_c \sim \mathrm{sgn}[\nabla_{x_c}\gamma(x_c)]v/\Delta$. Therefore $|\delta y_c|$ is larger for smaller $|\Delta|$ as shown in Fig. 5; the second gap is smaller and hence the wave packet in the second band shows larger $|\delta y_c/a|$.

Let us discuss an effect of disorder in real photonic crystals. It has been discussed that the photonic band with lower index is more robust against disorder [20]. Hence, in order to see the Hall effect of light, it is better to use the first band than the second band. While in Fig. 5 the shift of the first band is smaller than the second, it can be enhanced by reducing the gap between the two bands. The first band is not affected appreciably when fluctuations of system parameters are less than a few percent. For visible or near infrared light, this requires fabrication precision within ~ 50 nm, which is comparable to the current state of the art. Experimentally, the band splitting Δ of the order of 10^{-5} nm^{-1} can be controlled in the photonic crystal for the visible light [21]. This corresponds to $\delta y_c \sim 1/\Delta \sim 100a$. Thus, we can expect the shift as large as a hundred times the lattice constant a.

To conclude, we have derived the semiclassical equation of motion for the wave packet of light including the Berry curvature and resultant anomalous velocity. This gives a natural generalization of geometrical optics including the wave nature of light, and leads to the Hall effect of light. This also offers the natural framework to describe the ray of light in the photonic crystal without the inversion-symmetry where the Hall effect can be magnified tremendously.

Note added.—A related argument on a spin transport of photons came to our attention after the submission of this work. K. Yu. Bliokh and Yu. P. Bliokh, physics/0402110 (unpublished).

*Electronic address: m.onoda@aist.go.jp
†Electronic address: murakami@appi.t.u-tokyo.ac.jp
‡Electronic address: nagaosa@appi.t.u-tokyo.ac.jp

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