Fermionic Field Theory for Trees and Forests

Sergio Caracciolo,[1] Jesper Lykke Jacobsen,[2] Hubert Saleur,[3,4] Alan D. Sokal,[5] and Andrea Sportiello[1]
[1]*Dipartimento di Fisica, Università degli Studi di Milano and INFN, via Celoria 16, I-20133 Milano, Italy*
[2]*LPTMS, Université Paris-Sud, Bâtiment 100, F-91405 Orsay, France*
[3]*Service de Physique Théorique, CEA Saclay, F-91191 Gif-sur-Yvette, France*
[4]*Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089, USA*
[5]*Department of Physics, New York University, 4 Washington Place, New York, New York 10003, USA*
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We prove a generalization of Kirchhoff's matrix-tree theorem in which a large class of combinatorial objects are represented by non-Gaussian Grassmann integrals. As a special case, we show that unrooted spanning forests, which arise as a $q \to 0$ limit of the Potts model, can be represented by a Grassmann theory involving a Gaussian term and a particular bilocal four-fermion term. We show that this latter model can be mapped, to all orders in perturbation theory, onto the N-vector model at $N = -1$ or, equivalently, onto the σ model taking values in the unit supersphere in $\mathbb{R}^{1|2}$. It follows that, in two dimensions, this fermionic model is perturbatively asymptotically free.

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Kirchhoff's matrix-tree theorem [1] and its generalizations [2], which express the generating polynomials of spanning trees and rooted spanning forests in a graph as determinants associated to the graph's Laplacian matrix, play a central role in electrical-circuit theory [3] and in certain exactly soluble models in statistical mechanics [4,5]. Like all determinants, those arising in Kirchhoff's theorem can of course be rewritten as Gaussian integrals over fermionic (Grassmann) variables.

In this Letter, we prove a generalization of Kirchhoff's theorem in which a large class of combinatorial objects are represented by suitable non-Gaussian Grassmann integrals. Although these integrals can no longer be calculated in closed form, our identities allow the use of field-theoretic methods to shed new light on the critical behavior of the underlying geometrical models.

As a special case, we show that unrooted spanning forests, which arise as a $q \to 0$ limit of the q-state Potts model [6], can be represented by a Grassmann theory involving a Gaussian term and a particular bilocal four-fermion term. Furthermore, this latter model can be mapped, to all orders in perturbation theory, onto the N-vector model [O(N)-invariant σ model] at $N = -1$ or, equivalently, onto the σ model taking values in the unit supersphere in $\mathbb{R}^{1|2}$ [OSP(1|2)-invariant σ model]. It follows that, in two dimensions, this fermionic model is perturbatively asymptotically free, in close analogy to (large classes of) two-dimensional σ models and four-dimensional nonabelian gauge theories. Indeed, this fermionic model may, because of its great simplicity, be the most viable candidate for a rigorous nonperturbative proof of asymptotic freedom — a goal that has heretofore remained elusive in both σ models and gauge theories.

The plan of this Letter is as follows: First we prove some combinatorial identities involving Grassmann integrals, culminating in our general formula (12), and show how a special case yields unrooted spanning forests. Next we show that this latter model can be mapped onto the N-vector model at $N = -1$ and use this fact to deduce its renormalization-group (RG) flow at weak coupling. Finally, we conjecture the nonperturbative phase diagram in this model.

Combinatorial Identities.—Let $G = (V, E)$ be a finite undirected graph with vertex set V and edge set E. Associate to each edge e a weight w_e, which can be a real or complex number or, more generally, a formal algebraic variable. For $i \neq j$, let $w_{ij} = w_{ji}$ be the sum of w_e over all edges e that connect i to j. The (weighted) Laplacian matrix L for the graph G is then defined by $L_{ij} = -w_{ij}$ for $i \neq j$, and $L_{ii} = \sum_{k \neq i} w_{ik}$. This is a symmetric matrix with all row and column sums equal to zero.

Since L annihilates the vector with all entries 1, its determinant is zero. Kirchhoff's matrix-tree theorem [1] and its generalizations [2] express determinants of square submatrices of L as generating polynomials of spanning trees or rooted spanning forests in G. For any vertex $i \in V$, let $L(i)$ be the matrix obtained from L by deleting the ith row and column. Then Kirchhoff's theorem states that $\det L(i)$ is independent of i and equals

$$\det L(i) = \sum_{T \in \mathcal{T}} \prod_{e \in T} w_e, \tag{1}$$

where the sum runs over all spanning trees T in G. (We recall that a subgraph of G is called a tree if it is connected and contains no cycles, and is called spanning if its vertex set is exactly V.) The i-independence of $\det L(i)$ expresses, in electrical-circuit language, that it is physically irrelevant which vertex i is chosen to be "ground." There are many different proofs of Kirchhoff's formula (1); one simple proof is based on the Cauchy–Binet theorem in matrix theory (see, e.g., [7]).

More generally, for any sets of vertices $I, J \subseteq V$, let $L(I|J)$ be the matrix obtained from L by deleting the

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columns I and the rows J; when $I = J$, we write simply $L(I)$. The "principal-minors matrix-tree theorem" reads

$$\det L(i_1, \ldots, i_r) = \sum_{F \in \mathcal{F}(i_1,\ldots,i_r)} \prod_{e \in F} w_e, \tag{2}$$

where the sum runs over all spanning forests F in G composed of r disjoint trees, each of which contains exactly one of the "root" vertices $i_1, \ldots, i_r$. This theorem can easily be derived by applying Kirchhoff's theorem (1) to the graph in which the vertices $i_1, \ldots, i_r$ are contracted to a single vertex. Finally, the "all-minors matrix-tree theorem" (whose proof is more difficult; see [2]) states that, for any subsets I, J of the same cardinality r,

$$\det L(I|J) = \sum_{F \in \mathcal{F}(I|J)} \epsilon(F, I, J) \prod_{e \in F} w_e, \tag{3}$$

where the sum runs over all spanning forests F in G composed of r disjoint trees, each of which contains exactly one vertex from I and exactly one vertex (possibly the same one) from J; here $\epsilon(F, I, J) = \pm 1$ are signs whose precise definition is not needed here.

Let us now introduce, at each vertex $i \in V$, a pair of Grassmann variables $\psi_i, \bar{\psi}_i$. All of these variables are nilpotent ($\psi_i^2 = \bar{\psi}_i^2 = 0$), anticommute, and obey the usual rules for Grassmann integration [8]. Writing $\mathcal{D}(\psi, \bar{\psi}) = \prod_{i \in V} d\psi_i d\bar{\psi}_i$, we have, for any matrix A,

$$\int \mathcal{D}(\psi, \bar{\psi}) e^{\bar{\psi} A \psi} = \det A, \tag{4}$$

and, more generally,

$$\int \mathcal{D}(\psi, \bar{\psi}) \bar{\psi}_{i_1} \psi_{j_1} \cdots \bar{\psi}_{i_r} \psi_{j_r} e^{\bar{\psi} A \psi} = \epsilon(i_1, \ldots, i_r | j_1, \ldots, j_r) \det A(i_1, \ldots, i_r | j_1, \ldots, j_r), \tag{5}$$

where the sign $\epsilon(i_1, \ldots, i_r | j_1, \ldots, j_r) = \pm 1$ depends on how the vertices are ordered but is always $+1$ when $(i_1, \ldots, i_r) = (j_1, \ldots, j_r)$. These formulas allow us to rewrite the matrix-tree theorems in Grassmann form; for instance, (2) becomes

$$\int \mathcal{D}(\psi, \bar{\psi}) \left(\prod_{\alpha=1}^{r} \bar{\psi}_{i_\alpha} \psi_{i_\alpha} \right) e^{\bar{\psi} L \psi} = \sum_{F \in \mathcal{F}(i_1,\ldots,i_r)} \prod_{e \in F} w_e. \tag{6}$$

Let us now introduce, for each connected (not necessarily spanning) subgraph $\Gamma = (V_\Gamma, E_\Gamma)$ of G, the operator

$$Q_\Gamma = \left(\prod_{e \in E_\Gamma} w_e \right) \left(\prod_{i \in V_\Gamma} \bar{\psi}_i \psi_i \right). \tag{7}$$

(Note that each Q_Γ is even and hence commutes with the entire Grassmann algebra.) Now consider an unordered family $\mathbf{\Gamma} = \{\Gamma_1, \ldots, \Gamma_l\}$ with $l \geq 0$, and let us try to evaluate an expression of the form

$$\int \mathcal{D}(\psi, \bar{\psi}) Q_{\Gamma_1} \cdots Q_{\Gamma_l} e^{\bar{\psi} L \psi}. \tag{8}$$

If the subgraphs $\Gamma_1, \ldots, \Gamma_l$ have one or more vertices in common, then this integral vanishes on account of the nilpotency of the Grassmann variables. If, by contrast, the $\Gamma_1, \ldots, \Gamma_l$ are vertex-disjoint, then (6) expresses $\int \mathcal{D}(\psi, \bar{\psi}) (\prod_{k=1}^{l} \prod_{i \in V_{\Gamma_k}} \bar{\psi}_i \psi_i) e^{\bar{\psi} L \psi}$ as a sum over forests rooted at the vertices of $V_{\mathbf{\Gamma}} = \bigcup_{k=1}^{l} V_{\Gamma_k}$. In particular, all the edges of $E_{\mathbf{\Gamma}} = \bigcup_{k=1}^{l} E_{\Gamma_k}$ must be absent from these forests, since otherwise two or more of the root vertices would lie in the same component (or one of the root vertices would be connected to itself by a loop edge). On the other hand, by adjoining the edges of $E_{\mathbf{\Gamma}}$, these forests can be put into one-to-one correspondence with what we shall call $\mathbf{\Gamma}$-forests, namely, spanning subgraphs H in G whose edge set contains $E_{\mathbf{\Gamma}}$ and which, after deletion of the edges in $E_{\mathbf{\Gamma}}$, leaves a forest in which each tree component contains exactly one vertex from $V_{\mathbf{\Gamma}}$. (Equivalently, a $\mathbf{\Gamma}$-forest is a subgraph H with l connected components in which each component contains exactly one Γ_i, and which does not contain any cycles other than those lying entirely within the Γ_i. Note, in particular, that a $\mathbf{\Gamma}$-forest is a forest if and only if all the Γ_i are trees.) Furthermore, adjoining the edges of $E_{\mathbf{\Gamma}}$ provides precisely the factor $\prod_{e \in E_{\mathbf{\Gamma}}} w_e$. Therefore

$$\int \mathcal{D}(\psi, \bar{\psi}) Q_{\Gamma_1} \cdots Q_{\Gamma_l} e^{\bar{\psi} L \psi} = \sum_{H \in \mathcal{F}_{\mathbf{\Gamma}}} \prod_{e \in H} w_e, \tag{9}$$

where the sum runs over all $\mathbf{\Gamma}$-forests H.

We can now combine all the formulas (9) into a single generating function, by introducing a coupling constant t_Γ for each connected subgraph Γ of G. Since $1 + t_\Gamma Q_\Gamma = e^{t_\Gamma Q_\Gamma}$, we have

$$\int \mathcal{D}(\psi, \bar{\psi}) e^{\bar{\psi} L \psi + \sum_\Gamma t_\Gamma Q_\Gamma} = \sum_{\mathbf{\Gamma}\ \text{vertex-disjoint}} \left(\prod_{\Gamma \in \mathbf{\Gamma}} t_\Gamma \right) \sum_{H \in \mathcal{F}_{\mathbf{\Gamma}}} \prod_{e \in H} w_e. \tag{10}$$

We can express this in another way by interchanging the summations over $\mathbf{\Gamma}$ and H. Consider an arbitrary spanning subgraph H with connected components $H_1, \ldots, H_l$; let us say that Γ *marks* H_i (denoted $\Gamma \prec H_i$) in case H_i contains Γ and contains no cycles other than those lying entirely within Γ. Define the weight

$$W(H_i) = \sum_{\Gamma \prec H_i} t_\Gamma. \tag{11}$$

Then saying that H is a $\mathbf{\Gamma}$-forest is equivalent to saying that each of its components is marked by exactly one of the Γ_i; summing over the possible families $\mathbf{\Gamma}$, we obtain

$$\int \mathcal{D}(\psi, \bar{\psi}) e^{\bar{\psi} L \psi + \sum_\Gamma t_\Gamma Q_\Gamma} = \sum_{\substack{H\ \text{spanning} \subseteq G \\ H = (H_1, \ldots, H_l)}} \left[\prod_{i=1}^{l} W(H_i) \right] \prod_{e \in H} w_e. \tag{12}$$

This is our general combinatorial formula. Extensions allowing prefactors $\bar{\psi}_{i_1}\psi_{j_1}\cdots\bar{\psi}_{i_r}\psi_{j_r}$ are also easily derived.

We shall discuss elsewhere some of the applications of (12), and restrict attention here to the special case in which $t_\Gamma = t$ whenever Γ consists of a single vertex with no edges, $t_\Gamma = u$ whenever Γ consists of two vertices linked by a single edge, and $t_\Gamma = 0$ otherwise. We have

$$\int \mathcal{D}(\psi,\bar{\psi}) \exp\left[\bar{\psi}L\psi + t\sum_i \bar{\psi}_i\psi_i + u\sum_{\langle ij\rangle} w_{ij}\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j\right] = \sum_{\substack{F\in\mathcal{F}\\ F=(F_1,\ldots,F_l)}} \left[\prod_{i=1}^{l}(t|V_{F_i}| + u|E_{F_i}|)\right]\prod_{e\in F} w_e, \tag{13}$$

where the sum runs over spanning forests F in G with components $F_1,\ldots,F_l$; here $|V_{F_i}|$ and $|E_{F_i}|$ are, respectively, the numbers of vertices and edges in the tree F_i. [We remark that the four-fermion term $u\sum_{\langle ij\rangle} w_{ij}\bar{\psi}_i\psi_i\bar{\psi}_j\psi_j$ can equivalently be written, using nilpotency of the Grassmann variables, as $-(u/2)\sum_{i,j}\bar{\psi}_i\psi_i L_{ij}\bar{\psi}_j\psi_j$.] If $u = 0$, this formula represents vertex-weighted spanning forests as a massive fermionic free field [4,9]. More interestingly, since $|V_{F_i}| - |E_{F_i}| = 1$ for each tree F_i, we can take $u = -t$ and obtain the generating function of *unrooted* spanning forests with a weight t for each component. This is furthermore equivalent to giving each edge e a weight w_e/t and then multiplying by an overall prefactor $t^{|V|}$. This fermionic representation of unrooted spanning forests is the translation to generating functions and Grassmann variables of a little-known but important paper by Liu and Chow [10].

The generating function of unrooted spanning forests is also of interest because it arises as a $q \to 0$ limit of the q-state Potts model, in which the couplings $v_e = e^{\beta J_e} - 1$ tend to zero with fixed ratios $w_e = v_e/q$ [6].

Mapping onto lattice σ models.—We now claim that the model (13) with $u = -t$ can be mapped, to all orders in perturbation theory, onto the N-vector model at $N = -1$. Recall that the N-vector model consists of spins $\boldsymbol{\sigma}_i \in \mathbb{R}^N$, $|\boldsymbol{\sigma}_i| = 1$, located at the sites $i \in V$, with Boltzmann weight $e^{-\mathcal{H}}$, where $\mathcal{H} = -T^{-1}\sum_{\langle ij\rangle} w_{ij}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_i - 1)$ and T = temperature. Low-temperature perturbation theory is obtained by writing $\boldsymbol{\sigma}_i = (\sqrt{1 - T\boldsymbol{\pi}_i^2}, T^{1/2}\boldsymbol{\pi}_i)$ with $\boldsymbol{\pi}_i \in \mathbb{R}^{N-1}$ and expanding in powers of $\boldsymbol{\pi}$. Taking into account the Jacobian, the Boltzmann weight is $e^{-\mathcal{H}'}$, where

$$\begin{aligned}\mathcal{H}' &= \mathcal{H} + \frac{1}{2}\sum_i \log(1 - T\boldsymbol{\pi}_i^2)\\ &= \frac{1}{2}\sum_{i,j} L_{ij}\boldsymbol{\pi}_i\cdot\boldsymbol{\pi}_j - \frac{T}{2}\sum_i \boldsymbol{\pi}_i^2 - \frac{T}{4}\sum_{\langle ij\rangle} w_{ij}\boldsymbol{\pi}_i^2\boldsymbol{\pi}_j^2\\ &\quad + O(\boldsymbol{\pi}_i^4, \boldsymbol{\pi}_j^4).\end{aligned} \tag{14}$$

When $N = -1$, the bosonic field $\boldsymbol{\pi}$ has -2 components and therefore can be replaced by a fermion pair $\psi, \bar{\psi}$ if we make the substitution $\boldsymbol{\pi}_i\cdot\boldsymbol{\pi}_j \to \psi_i\bar{\psi}_j - \bar{\psi}_i\psi_j$. Higher powers of $\boldsymbol{\pi}_i^2$ vanish due to the nilpotence of the Grassmann fields, and we obtain the model (13) if we identify $t = -T$, $u = T$. Note the reversed sign of the coupling: the spanning-forest model with positive weights ($t > 0$) corresponds to the *antiferromagnetic* N-vector model ($T < 0$).

An alternate mapping can be obtained by introducing at each site, in addition to the Grassmann fields $\psi_i, \bar{\psi}_i$, an auxiliary one-component bosonic field φ_i satisfying the constraint $\varphi_i^2 + 2t\bar{\psi}_i\psi_i = 1$. Solving this constraint yields $\varphi_i = 1 - t\bar{\psi}_i\psi_i = e^{-t\bar{\psi}_i\psi_i}$ and

$$\begin{aligned}\delta(\varphi_i^2 + 2t\bar{\psi}_i\psi_i - 1) &= \frac{1}{2\varphi_i}\delta(\varphi_i - (1 - t\bar{\psi}_i\psi_i))\\ &= \frac{e^{t\bar{\psi}_i\psi_i}}{2}\delta(\varphi_i - (1 - t\bar{\psi}_i\psi_i)).\end{aligned} \tag{15}$$

If we define the superfield $\vec{\boldsymbol{\sigma}}_i = (\varphi_i, \psi_i, \bar{\psi}_i)$ with inner product $\vec{\boldsymbol{\sigma}}_i\cdot\vec{\boldsymbol{\sigma}}_j = \varphi_i\varphi_j + t(\bar{\psi}_i\psi_j - \psi_i\bar{\psi}_j)$, then the σ model with Hamiltonian $\mathcal{H} = -T^{-1}\sum_{\langle ij\rangle} w_{ij}(\vec{\boldsymbol{\sigma}}_i\cdot\vec{\boldsymbol{\sigma}}_j - 1)$ and constraint $\vec{\boldsymbol{\sigma}}_i\cdot\vec{\boldsymbol{\sigma}}_i = 1$ corresponds to the fermionic model (13) if we again make the identification $t = -T$, $u = T$. This σ model, which is invariant under the supergroup OSP(1|2), has been studied previously by one of us [11]. It is presumably nonperturbatively equivalent to the N-vector model at $N = -1$, on the grounds that each fermion equals -1 boson.

It is worth mentioning that the correspondence between the spanning-forest model and these two σ models, while valid at all orders of perturbation theory, does *not* hold nonperturbatively. (This can be checked explicitly in the exact solution for the two-site model [12].) The error arises from neglecting the second square root when solving the constraints; we did not, in fact, parametrize a (super)sphere but rather a (super)-hemisphere. Indeed, since $t > 0$ corresponds to an antiferromagnetic σ model, the terms we have neglected are actually dominant. But no matter: the perturbative correspondence is still correct, and has the RG consequences discussed below. Furthermore, we conjecture that a nonperturbative correspondence can be obtained by using a σ model with a suitable variant Boltzmann weight.

Continuum limit.—Suppose now that the graph G is a regular two-dimensional lattice, with weight $w_{ij} = w > 0$ for each nearest-neighbor pair. We can then read off, from known results on the N-vector model [13], the RG flow for the spanning-forest model: it is

$$\frac{d\bar{t}}{d\ell} = \frac{3}{2\pi}\bar{t}^2 - \frac{3}{(2\pi)^2}\bar{t}^3 + \frac{2.342\,784\,57}{(2\pi)^3}\bar{t}^4 + \frac{1.436\,77}{(2\pi)^4}\bar{t}^5 + \cdots, \tag{16}$$

where $\bar{t} = t/w$, and ℓ is the logarithm of the length rescaling factor; here the first two coefficients are universal (after suitable normalization of the kinetic term), while the remaining coefficients are for the square lattice only. The positive coefficient of the $\bar{t}^2$ term indicates that for $t > 0$ the model is perturbatively asymptotically free. Indeed, two-dimensional N-vector models are asymptotically free for the usual sign of the coupling ($T > 0$) when $N > 2$, but for the reversed sign of the coupling ($T < 0$) when $N < 2$. Assuming that the asymptotic freedom holds also nonperturbatively, we conclude that for $t > 0$ the model is attracted to the infinite-temperature fixed point at $t = +\infty$, hence is massive and OSP(1|2) symmetric. For $t_{\rm crit} < t < 0$, by contrast, the model is attracted to the free-fermion fixed point at $t = 0$, and hence is massless with central charge $c = -2$, with the OSP(1|2) symmetry spontaneously broken. Finally, for $t < t_{\rm crit}$ we expect that the model will again be massive, with the OSP(1|2) symmetry restored.

More specifically, for $t > 0$ it is predicted that the correlation length diverges for $t \downarrow 0$ (or $w \uparrow +\infty$) as

$$\xi = C_\xi e^{(2\pi/3)(w/t)}\left(\frac{2\pi}{3}\frac{w}{t}\right)^{1/3} \times \left[1 - 0.011\,622\,120\,4\frac{t}{w} + 0.004\,461\,42\frac{t^2}{w^2} + \cdots\right], \tag{17}$$

where C_ξ is a nonperturbative constant (the terms in brackets are for the square lattice only). The numerical results of [6], based on transfer matrices and finite-size scaling, are consistent with the nonperturbative validity of the asymptotic-freedom predictions (16) and (17), but are inconclusive because the strip widths are small ($L \le 10$). It would be interesting to make a Monte Carlo test of (17), at large correlation lengths, along the lines of [14].

The numerics of [6] are also consistent with the central charge $c = -2$ in the massless phase $t_{\rm crit} < t < 0$ but are not definitive because of the strong ($1/\log$) corrections to scaling induced by the marginally irrelevant operator.

Finally, the critical point $t_{\rm crit}$ presumably corresponds to the $q \to 0$ limit of the antiferromagnetic critical curve in the q-state Potts model, under the identification $w/t = (e^{\beta J} - 1)/q$. Known exact results for the square lattice [15,16] yield $(w/t)_{\rm crit} = -1/4$. The analysis of the critical theory proves rather difficult, but there are strong indications that it is simply a free OSP(1|2) model, i.e., the theory of a noncompact boson and a pair of fermions, with central charge $c = -1$.

Let us also remark that there exists a much-studied variant of the N-vector model in which the high-temperature expansion on the lattice has been truncated so as to forbid loop crossings [17]. For $-2 < N < 2$, this model possesses several critical points; in particular, the dilute-loop critical point is expected to be generic in the sense that adding loop crossings acts as an irrelevant perturbation. For $N = -1$, this yields a $c = -3/5$ theory [18]; the relation to the $c = -1$ theory discussed above is mysterious and deserves further study.

It would also be interesting to know whether our identities are in any way related to the forest-root formula of Brydges and Imbrie [19], which leads to a dimensional-reduction formula for branched polymers.

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