Will a Large Complex System with Time Delays Be Stable?

Viktor K. Jirsa and Mingzhou Ding

Center for Complex Systems and Brain Sciences, Florida Atlantic University, Boca Raton, Florida 33431, USA
(Received 9 February 2004; published 12 August 2004)

In 1972 May showed that for a large linear system with random coupling the system size and the average coupling strength must together satisfy a simple inequality to ensure the stability of the equilibrium point. Here we extend the analysis to delay coupled systems. Our calculations establish that the same inequality obtained by May constrains the stability for systems randomly coupled through discrete and distributed delays.

Introduction.—Large complex dynamical systems with many interacting components arise in many areas of science and engineering including physics, biology, and ecology. A property of essential importance in such systems is the stability of certain synchronized dynamical states. Naively, one expects that increasing the number of interacting elements or increasing the strength of interaction will enhance the system's stability. In a now classic paper [1] entitled "Will a Large Complex System Be Stable?" May examined this issue in a linear system with random coupling. Using results from the random matrix theory [2,3] he showed that the system size and the coupling strength must together satisfy a simple inequality. A system will become unstable if there are too many elements or if the interactions are too strong. Some refinements of May's result have appeared in the literature since 1972 [4], but the essential idea contained in May's paper remains intact today.

The past few decades have witnessed large strides of progress in nonlinear dynamical systems theory and in systems with transmission time delays. Recent work has begun to study the stability of synchronized dynamics in randomly coupled nonlinear systems [5–7]. In this Letter we consider the stability of large systems with time delayed random coupling. Our main result is the derivation of the exact stability boundary in the parameter space which, when combined with the random matrix theory, allows us to address the question given in the title of this Letter.

Background.—Consider an N-dimensional state vector $\mathbf{x} = (\cdots x_i \cdots)$ with $x_i \in \mathcal{R}$ and $i = 1, \ldots, N$. The state vector $\mathbf{x}$ is considered a perturbation to a fixed point solution of a general system of nonlinear differential equations. To study the stability of the fixed point solution, let the linearized equations of motion in component form be

$$\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} a_{ij} x_j(t), \qquad (1)$$

where $a_{ij} \in \mathcal{R}$ is the coupling strength. Letting $\mathbf{A} = (a_{ij})$ be the coupling matrix, (1) can be written as

$$\dot{\mathbf{x}}(t) = -\mathbf{x}(t) + \mathbf{x}(t)\mathbf{A}^T, \qquad (2)$$

where T stands for matrix transpose. Decompose the coupling matrix according to $\mathbf{A}^T = \mathbf{E}\Lambda\mathbf{E}^{-1}$, where Λ is the Jordan form with complex eigenvalues $\lambda \in C$ and $\mathbf{E}$ contains the corresponding eigenvectors $\mathbf{e}$. Multiplying (2) from the right with $\mathbf{E}$ we obtain

$$d\mathbf{x}(t)\mathbf{E}/dt = -\mathbf{x}(t)\mathbf{E} + \mathbf{x}(t)\mathbf{E}\Lambda. \qquad (3)$$

This leads to a decoupled representation of the dynamics of (2) in terms of its eigenmodes

$$\dot{u}(t) = -u(t) + \lambda u(t), \qquad \lambda \in C, \qquad (4)$$

where $u = \mathbf{x}\mathbf{e}$. Hence we reduced the discussion of the stability of the N-dimensional system in (2) to the study of the stability of uncoupled eigenmode equations corresponding to all the distinct eigenvalues λ of $\mathbf{A}^T$. Let the eigenvalue $\lambda = a + ib$, $a, b \in \mathcal{R}$. Clearly, for the eigenmode to be stable we must have $\text{Re}(\lambda) = a < 1$. This stability condition gives a stability region in the complex λ plane as shown in Fig. 1.

To study the stability of Eq. (2) as a function of system size N and average coupling strength we follow the formulation of May [1]. Let $a_{ij} = r_{ij} b_{ij}$. Here the variable r_{ij}
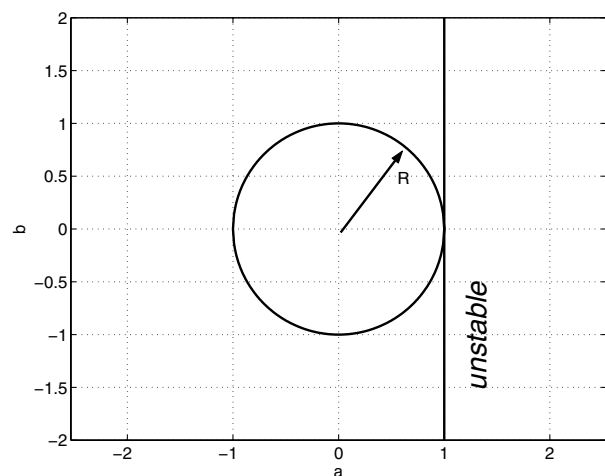

FIG. 1. The stability region of Eq. (4), defined by $\text{Re}(\lambda) = a < 1$, is the region to the left of the vertical line. If all the eigenvalues of the coupling matrix $\mathbf{A}$ are contained in the unit disk of radius $R = 1$, then the system (2) is stable.

describes the connectivity of the coupled system. Specifically, we let $r_{ij} = 1$ with probability $0 \leq C \leq 1$ and $r_{ij} = 0$ with probability $1 - C$. The variables b_{ij} are independent identically distributed Gaussian random variables with zero mean and standard deviation σ. Clearly, a_{ij} has zero mean and standard deviation $\sqrt{C}\sigma$ which measures the magnitude of average coupling strength between the elements in the system. The coupling matrix $\mathbf{A}^T$ is typically asymmetric. According to the theory of random matrices [2,3], for large N, the eigenvalues of the coupling matrix $\mathbf{A}$ are contained in the disk of radius $R = \sqrt{NC}\sigma$ which is centered at the origin. If this disk is contained in the stability region shown in Fig. 1 then the system is stable. Otherwise it is unstable. This idea leads to the stability condition

$$R = \sqrt{NC}\sigma < 1, \qquad (5)$$

which was obtained first by May in 1972 [1].

The case of discrete delay.—We consider the effect of discrete time delay on the above stability results. Let the equations of motion be

$$\dot{\mathbf{x}}(t) = -\mathbf{x}(t) + \mathbf{x}(t - \tau)\mathbf{A}^T. \qquad (6)$$

The decoupled eigenmode equation is

$$\dot{u}(t) = -u(t) + \lambda u(t - \tau), \qquad \lambda \in C. \qquad (7)$$

The stability of the delay differential Eq. (7) is determined by the characteristic equation

$$H(z) = z + 1 - \lambda e^{-z\tau} = 0, \qquad (8)$$

where $u(t) = e^{zt}$, $z \in C$. If all the roots satisfy $\mathrm{Re}(z) < 0$ then the solution is stable. It is easy to see that, for $\mathrm{Re}(\lambda) = a = 1$ and $\mathrm{Im}(\lambda) = b = 0$, we have $z = 0$ which corresponds to a change of stability. This point ($a = 1$ and $b = 0$) in the λ plane is also part of the boundary separating stable from unstable regions in Fig. 1 for the $\tau = 0$ undelayed system. This fact is significant since it means that regardless of the delay τ the stability condition for Eq. (6) is at most the unit circle as in Eq. (5). What is not known at this stage is what happens to the other part of the stability region in Fig. 1 as τ increases.

The possibility of a change of sign of $\mathrm{Re}(z)$ by way of $\mathrm{Re}(z) = \infty$ is excluded by a theorem of Datko [8]. Hence all other sign changes of $\mathrm{Re}(z)$ must occur at purely imaginary $z = i\omega$, $\omega \in \mathcal{R}_0^+$. We construct the critical surface in the space spanned by the three parameters (a, b, τ) which identifies boundaries of the stability region. Inserting $z = i\omega$ into (8) we obtain after some manipulations

$$\tan\omega\tau_c = \frac{b - a\omega}{a + b\omega}, \qquad (9)$$

which provides the condition for the critical delay $\tau = \tau_c$ at which a change in stability may occur and

$$\omega^2 + 1 = \lambda^2 = a^2 + b^2, \qquad (10)$$

which identifies the critical frequency. From these two equations the critical surface $\tau_c = g(a, b)$ is given by

$$\tau_c = \frac{-1}{\sqrt{|\lambda|^2 - 1}} \tan^{-1} \frac{a\sqrt{|\lambda|^2 - 1} - b}{a + b\sqrt{|\lambda|^2 - 1}}. \qquad (11)$$

Similar results given implicitly in polar coordinates may be found in [9]. Our result is illustrated in Fig. 2. The half volume $a > 1$ contains solutions which are always unstable. If $a < 1$, then the cylinder defined by $|\lambda|^2 = a^2 + b^2 = 1$ contains solutions which are stable for all delays τ. Outside of this cylinder the size of the stability region depends on τ. Cross sections for various values of τ are shown in Fig. 3. For $\tau = 0$, the stability region is the entire half plane $a < 1$. For nonzero τ, the stability regime is finite and teardrop shaped. The stability region decreases and approaches the stability cylinder defined by $a^2 + b^2 = 1$ for increasing values of τ. The nature of the stability change for increasing τ is determined uniquely by the sign of $d\mathrm{Re}(z)/d\tau$ [10] which can be written as

$$\frac{d\mathrm{Re}(z)}{d\tau} = -\mathrm{Re}\left[\frac{\partial H/\partial\tau}{\partial H/\partial z}\right] = \omega^2/|D|^2 > 0, \qquad (12)$$

where $D = e^{z\tau} + \lambda\tau$. This means that as τ increases across the critical surface from below, the system always becomes unstable.

The case of distributed delay.—Here we consider the effect of distributed delay on the stability results above. Let the equations of motion be

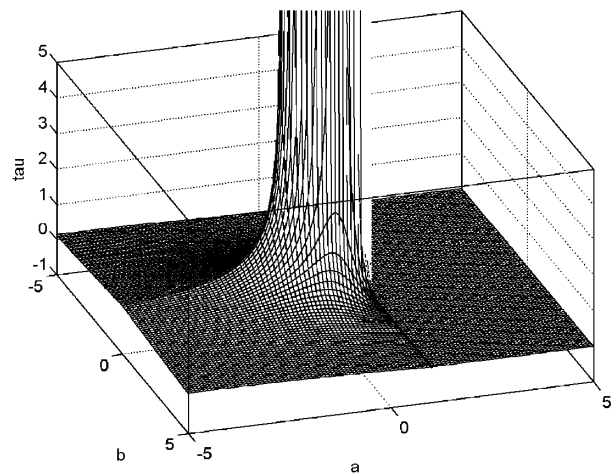$$\dot{x}_i = -x_i + \sum_{j=1}^{N} a_{ij} \int_0^\infty f(\tau', d)x_j(t - \tau')d\tau'. \qquad (13)$$

FIG. 2. The critical surface given in (11) defines the minimal critical delay values τ_c at which the system (1) becomes unstable. For $a > 1$, all solutions are unstable. For $a < 1$, only the solutions below the critical surface, $\tau < \tau_c$, are stable. All solutions above the critical surface are unstable.
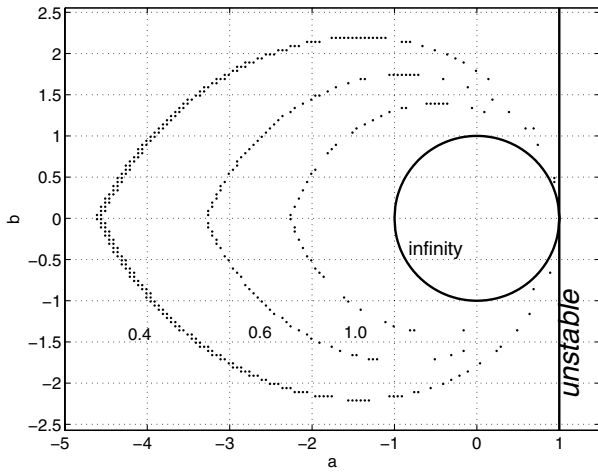
FIG. 3. Cross sections of the critical surface defined by (11) are shown for the delay values $\tau = 0, 0.4, 0.6, 1.0$, and ∞. For $\tau = 0$, the entire left half plane, $a < 1$, is the stability region, whereas for $\tau = \infty$ it is the unit circle centered at the origin. For intermediate values, the stability region decreases as τ increases.

The real-valued kernel function $f(\tau', d) = f(\tau', \tau, d)$ with mean delay $\tau \in \mathcal{R}_0^+$ captures the continuous distribution of delay values and reduces to Dirac's delta function $\delta(\tau' - \tau)$ when its width d approaches zero. We require normalization $\int_0^\infty f(\tau', d)d\tau' = 1$ and impose the condition that $\tau \gg d$. In addition causality requires that $f(\tau', d) = 0$ for all $\tau' < 0$. These constraints on the distribution of the delays are satisfied in most applications. In matrix form Eq. (13) is

$$\dot{\mathbf{x}} = -\mathbf{x} + \int_0^\infty f(\tau', d)\mathbf{x}(t - \tau')d\tau'\mathbf{A}^T. \quad (14)$$

The eigenmode mode equation is

$$\dot{u} = -u + \lambda \int_0^\infty f(\tau', d)u(t - \tau')d\tau', \qquad \lambda \in C. \quad (15)$$

The stability of the integral differential Eq. (15) is de-

termined by the characteristic equation

$$H(z) = z + 1 - \lambda e^{-z\tau}F(z) = 0, \quad (16)$$

where the function $F(z) = F(z, d)$ is defined as $F(z) = \int_{-\infty}^\infty \tilde{f}(\tau', d)e^{-z\tau'}d\tau'$ with $\tau \gg d$ and $\tilde{f}(\tau', d) = f(\tau + \tau', d)$. If all the roots satisfy $\text{Re}(z) < 0$, where $u = e^z, z \in C$ then the solution of (13) will be stable. It is important to note that, for Eq. (16), setting $a = 1$ and $b = 0$ again leads to $z = 0$ which corresponds to a change of stability. Thus, the disk of stability can be at most the unit circle. To investigate the critical surface we let $z = i\omega, \omega \in \mathcal{R}_0^+$. In this case, $F(\omega) = F_1(\omega) + iF_2(\omega)$ becomes identical with the complex-valued Fourier transform of the kernel function $\tilde{f}(\tau', d)$. We insert $z = i\omega$ into (16) and obtain

$$\Gamma(\omega) = \tan\omega\tau_c + \frac{F_1(\omega)(a\omega - b) - F_2(\omega)(a + b\omega)}{F_1(\omega)(a + b\omega) + F_2(\omega)(a\omega - b)} = 0 \quad (17)$$

and

$$\omega^2 + 1 = |\lambda|^2|F(\omega)|^2. \quad (18)$$

Solving for ω we have an equation that determines the critical surface in the (a, b, τ) space. Unfortunately, for a general delay kernel, we are not able to obtain explicit formulas for the critical surface. However, it turns out that all positive definite and normalizable functions $f(\tau', d)$ (given a mild sufficient condition) result in a critical surface which is bounded from below by the critical surface of the discrete delay case. As a consequence, the discrete delay ($d = 0$) is the most destabilizing case. In other words, the system becomes less unstable as the width d of the time delay distribution increases. This can be understood as follows: The frequency ω is obtained from (18) as the intersection of the two curves $y_1(\omega) = \omega^2 + 1$ and $y_2(\omega) = |\lambda|^2|F(\omega)|^2$. In particular, $F(\omega)$ has the property $|F(\omega = 0)| = |\int_{-\infty}^\infty \tilde{f}(\tau', d)d\tau'| = |\int_{-\infty}^\infty f(\tau', d)d\tau'| = 1$ and

$$(19)$$

$$|F(\omega \neq 0)| = \left| \int_{-\infty}^\infty \tilde{f}(\tau', d)e^{-i\omega\tau'}d\tau' \right| \leq \int_{-\infty}^\infty |\tilde{f}(\tau', d)|\underbrace{|e^{-i\omega\tau'}|}_{=1}d\tau' \leq \int_{-\infty}^\infty f(\tau', d)d\tau' = 1.$$

The equality sign holds for the discrete delay case, $\lim_{d\to 0} F(\omega, d) = 1 \; \forall \; \omega$. Taken together with the fact that $y_1(\omega)$ is a positive definite and monotonically increasing curve, it follows that the discrete delay case results in an intersection of the two curves $y_1(\omega)$ and $y_2(\omega) = |\lambda|^2$ at the highest frequency ω. Every other distribution function will have its intersection of $y_1(\omega)$ and $y_2(\omega) = |\lambda|^2|F(\omega)|^2 < |\lambda|^2$ at a smaller frequency ω. To understand the effects of the decreased frequency on the critical surface, we study $d\tau/d\omega$ and consider first a distribution function $f(\tau', d)$ symmetric around τ. In this

case the Fourier transform $F(\omega) = F_1(\omega)$ is real valued with $F_2(\omega) = 0$. Then Eq. (17) becomes identical to (9) as discussed in the discrete delay case. We characterize the change of the critical surface by

$$\frac{d\tau}{d\omega} = -\frac{\partial\Gamma/\partial\omega}{\partial\Gamma/\partial\tau} = -\tau/\omega - \frac{(a^2 + b^2)\cos^2\omega\tau}{\omega(a + b\omega)^2} < 0 \quad (20)$$

for all $\omega, \tau \in \mathcal{R}_0^+$. As a consequence, any decrease in ω results in an increase of the stability region in the com-
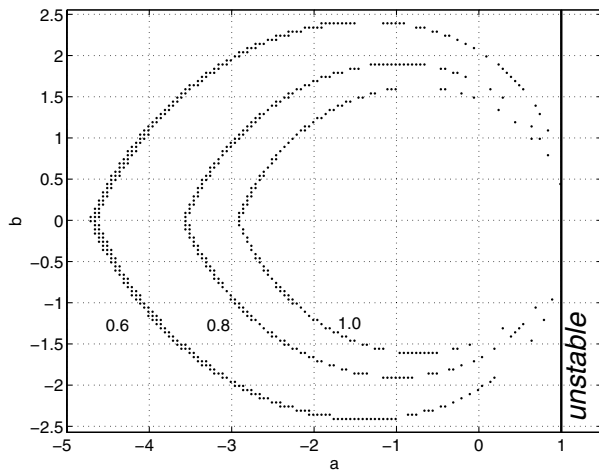
FIG. 4. Cross sections of the critical surface for the distributed delay case are shown for the values $\omega = 1, 0.8$, and 0.6 at a time delay of $\tau = 0.7$. It is evident that the stability area increases with decreasing ω.

plex λ plane and hence a stabilization of the system (13) compared to the discrete delay case. This general result is illustrated in Fig. 4 for symmetric distribution functions $f(\tau', d)$. For arbitrary distribution functions, $F(\omega) = F_1(\omega) + iF_2(\omega)$ is generally complex valued and results in a rotation of the critical surface around the vertical axis at $a = 1$ and $b = 0$ as described by (17). In this case, a lengthy calculation shows that all our results remain valid for general distribution functions if $F_1'(\omega)F_2(\omega) - F_2'(\omega)F_1(\omega) \geq 0$ holds, where the prime denotes the derivative with respect to ω. Under this sufficient condition, the critical surface for the discrete delay case defines the lower bound for the critical surfaces of arbitrary distribution functions.

Finally, the nature of a stability change for increasing τ is determined uniquely by the sign of $d\mathrm{Re}(z)/d\tau$ [10] which is obtained from (16)

$$\frac{d\mathrm{Re}(z)}{d\tau} = -\mathrm{Re}\left[\frac{\partial H/\partial \tau}{\partial H/\partial z}\right]. \qquad (21)$$

After some algebra we have

$$\frac{d\mathrm{Re}(z)}{d\tau} = \frac{\omega^2}{|D|^2} - \frac{\omega(\omega^2+1)}{|D|^2}\mathrm{Re}\left[\frac{\partial}{\partial \omega}\ln F(\omega)\right], \quad (22)$$

where

$$D = 1 + \lambda\tau e^{-z\tau}F(z) - \lambda\tau e^{-z\tau}F'(z) \qquad (23)$$

and $F'(z) = \partial F(z)/\partial z$. If $d\mathrm{Re}(z)/d\tau$ is greater than zero, then the system becomes unstable as τ is increased from below the critical surface to above it. Otherwise we have a stabilizing bifurcation.

For illustration we consider two specific cases to generate some insights into the problem. In one example

the kernel function is taken to be the Gaussian function $f(\tau', d) = (2\pi d^2)^{-0.5}e^{-(\tau'-\tau)^2/(2d^2)}$. The other example is the uniform distribution $f(\tau', d) = (2d)^{-1}$ for $-d \leq \tau' - \tau \leq d$ and zero otherwise. Since both distributions are symmetric, we may apply (9) to identify the critical surface parametrized by ω. The frequency is obtained from (18) as $\omega^2 = (|\lambda|^2 - 1)(1 + d|\lambda|^2)^{-1}$ for the Gaussian distribution, and $\omega^2 = (|\lambda|^2 - 1) \times (1 + d^2|\lambda|^2/3)^{-1}$ for the uniform distribution for small d. Both distributions result in smaller frequencies ω for increasing width d and hence in greater critical delays than the discrete delay case. As a consequence, their critical surfaces are bounded from below by the critical surface of the discrete delay case. The nature of the instability at the critical surface is destabilizing for both distributions: for the Gaussian distribution function, it follows from (22) that $d\mathrm{Re}(z)/d\tau = \omega^2|D|^{-2}[1 + d(\omega^2 + 1)] > 0$ and hence results in destabilization. Note that D has been defined in (23). The uniform distribution results in destabilization as τ increases through the critical surface since $d\mathrm{Re}(z)/d\tau = |D|^{-2}[\omega^2 + (\omega^2 + 1) \times (1 - \omega d/\tan\omega d)] > 0$ for $\omega d \in [0, \pi/2]$.

Discussion.—Now we are ready to tackle the title question. For both discrete and distributed delay cases, the point $a = 1$ and $b = 0$ separates stability from instability. This means that the stability disk for the random coupling matrix is at most the unit circle. On the other hand, it is clear that, for the discrete delay case, the unit circle remains stable for all delays. The distributed delay results in critical surfaces that lie above the critical surface of the discrete delay case. This means that the unit circle is again stable for typical delay distribution functions. These observations lead to the remarkable result that May's stability condition Eq. (5) remains intact despite the fact that the introduction of time delay clearly reduces the regions of stability.

[1] R. M. May, Nature (London) **238**, 413 (1972).
[2] M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991).
[3] V. L. Girko, Theory Probab. Appl. **29**, 694 (1984).
[4] J. E. Cohen and C. M. Newman, Ann. Probab. **12**, 283 (1984).
[5] P. M. Gade, Phys. Rev. E **54**, 64 (1996).
[6] M. Timme, F. Wolf, and T. Geisel, Phys. Rev. Lett. **92**, 074101 (2004).
[7] Y. Deng, M. Ding, and J. Feng, J. Phys. A **37**, 2163 (2004).
[8] R. Datko, Q. Appl. Math. **36**, 279 (1978).
[9] R. D. Braddock and P. Van den Driessche, J. Aust. Math. Soc. Ser. B, Appl. Math. **19**, 358 (1976).
[10] K. L. Cooke and Z. Grossman, J. Math. Anal. Appl. **86**, 592 (1982).