Comment on "Steep Sharp-Crested Gravity Waves on Deep Water"

Lukomsky *et al.* [1] give numerical evidence for the existence of a family of two-dimensional irrotational symmetric periodic gravity waves having a stagnation point of the flow field inside the fluid domain and for which the horizontal water velocities near the crest exceed the wave speed. In our opinion, the approach and the conclusions of Ref. [1] are wrong, and here we present our arguments.

In essence, Ref. [1] supports its claim by using truncated Fourier series to approximate the flow. The convergence issue is not addressed. This failure is the cause of the erroneous conclusions of the Letter.

Using the notation in [1], if (1) holds in the distribution sense, classical elliptic theory [2] shows that the velocity potential $\Phi(\theta, y)$ is smooth in the fluid region $\Omega = \{(\theta, y): x \in R, y < \eta(\theta)\}$. But then the stream function $\Psi(\theta, y)$, of which $\Phi(\theta, y)$ is the harmonic conjugate in Ω, is also smooth in Ω. The free surface of the irregular wave described in [1] is not the graph of a continuous function. However, every component of the complement of Ω consists of more than a single point since we do not have points on the boundary of Ω for which there is a closed curve surrounding them and consisting of points of Ω only. Also, note that the free surface is a streamline, that is, a level set of the function $[\Psi(\theta, y) - cy]$. Therefore, by the Perron method (see [2] or [3]), the harmonic function $[\Psi(\theta, y) - cy]$ is continuous up to the boundary of the fluid domain Ω.

The "irregular wave" described in [1] has one stagnation point located at a point O_1 below the crest on the axis of symmetry of the wave. Therefore, the smoothness of Ψ yields by the implicit function theorem that all streamlines not passing through O_1 are smooth curves within Ω. Also, note that by Bernoulli's Law and condition (2), the velocity field (Φ_θ, Φ_y) is bounded on compact subsets of the closure of Ω. Consider the "critical streamline" passing through the stagnation point O_1 and consisting of two symmetric curves intersecting at O_1, as depicted in Fig. 3 from Ref. [1]. Any streamline passing through a point in Ω located above the critical streamline cannot cross the critical streamline by the uniqueness property of the differential system for the particle trajectories, ensured by the smoothness of Ψ in Ω [4]. Moreover, the equilibrium point O_1, being a saddle point in the phase plane of the differential system for the particle trajectories, cannot be a limit point. We conclude by the Poincaré-Bendixon theorem [4] that each such streamline must intersect the free surface since there are no equilibrium points above the critical streamline. But then $[\Psi(\theta, y) - cy]$ has to be equal to its constant value on the free surface in any fluid region bounded below by the critical streamline. This means that there is a whole layer of stagnation points of the flow, not just O_1. The obtained contradiction proves the impossibility of the occurrence of the wave pattern claimed in Ref. [1].

We conclude with a remark. Numerical simulations in [5,6] indicate that periodic gravity waves with stagnation points of the flow inside the fluid domain (and with overhanging profiles) are possible for water flows with constant vorticity. See [7–9] for recent results on the existence of rotational periodic gravity waves approaching flows with stagnation points.

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