Multifractal Statistics of Lagrangian Velocity and Acceleration in Turbulence

L. Biferale,[1] G. Boffetta,[2,6] A. Celani,[3] B. J. Devenish,[1,7] A. Lanotte,[4] and F. Toschi[5]

[1]*Department of Physics, University of Rome "Tor Vergata," Via Ricerca Scientifica 1, 00133 Roma, Italy*
[2]*Department of Physics and INFM, University of Torino, Via Pietro Giuria 1, 10125, Torino, Italy*
[3]*CNRS-INLN, 1361 Route des Lucioles, 06560 Valbonne, France*
[4]*CNR-ISAC (Sezione di Lecce), Strada Provinciale Lecce-Monteroni, 73100 Lecce, Italy*
[5]*Istituto per le Applicazioni del Calcolo, CNR, Viale del Policlinico 137, 00161 Roma, Italy*
[6]*CNR-ISAC (Sezione di Torino), Corso Fiume 4, 10133 Torino, Italy*
[7]*The Met Office, Fitzroy Road, Exeter, Devon EX1 3PB, United Kingdom*
(Received 9 March 2004; published 4 August 2004)

The statistical properties of velocity and acceleration fields along the trajectories of fluid particles transported by a fully developed turbulent flow are investigated by means of high resolution direct numerical simulations. We present results for Lagrangian velocity structure functions, the acceleration probability density function, and the acceleration variance conditioned on the instantaneous velocity. These are compared with predictions of the multifractal formalism, and its merits and limitations are discussed.

DOI: 10.1103/PhysRevLett.93.064502 PACS numbers: 47.27.–i, 47.10.+g

Understanding the Lagrangian statistics of particles advected by a turbulent velocity field, $\boldsymbol{u}(\boldsymbol{x}, t)$, is important both for its theoretical implications [1] and for applications, such as the development of phenomenological and stochastic models for turbulent mixing [2]. Recently, several authors have attempted to describe Lagrangian statistics such as acceleration by constructing models based on equilibrium statistics (see, e.g., [3–5], critically reviewed in [6]). In this Letter we show how the multifractal formalism offers an alternative approach which is rooted in the phenomenology of turbulence. Here, we propose a derivation of the Lagrangian statistics directly from the Eulerian statistics.

In order to obtain an accurate description of the particle statistics it is necessary to measure the positions, $\boldsymbol{X}(t)$, and velocities, $\boldsymbol{v}(t) \equiv \dot{\mathbf{X}}(t) = \boldsymbol{u}(\mathbf{X}(t), t)$, of the particles with very high resolution, ranging from fractions of the Kolmogorov time scale, τ_η, to multiples of the Lagrangian integral time scale, T_L. The ratio of these time scales, T_L/τ_η, gives an estimate of the microscale Reynolds number, R_λ, which may easily reach values of order 10^3 in laboratory experiments. Despite recent advances in experimental techniques for measuring Lagrangian turbulent statistics [7–9], direct numerical simulations (DNS) still offer higher accuracy albeit at a slightly lower Reynolds number [10–13]. In this Letter we are concerned with single particle statistics, that is, the statistics of velocity and acceleration fluctuations along individual particle trajectories. Here, we analyze Lagrangian data obtained from a recent DNS of homogeneous isotropic turbulence [14] which was performed on 512^3 and 1024^3 cubic lattices with Reynolds numbers up to $R_\lambda \sim 280$. The Navier-Stokes equations were integrated using fully dealiased pseudospectral methods for a total time $T \approx T_L$. The flow was forced by keeping the total energy constant in the first two wave number shells (for more details, see [14]). Approximately 2×10^6 Lagrangian particles (passive tracers) were released into the flow once a statistically stationary velocity field had been obtained. The positions and velocities of the particles were stored at a sampling rate of $0.07\tau_\eta$. The Lagrangian velocity was calculated using linear interpolation. Acceleration was calculated both by following the particle and by direct computation from all three forces acting on the particle—the pressure gradients, viscous forces, and the large scale forcing. The two measurements were found to be in very good agreement. The Lagrangian statistics were calculated by averaging over all particle trajectories and over all time.

It is well known that Lagrangian velocity increments, $\delta_\tau \boldsymbol{v} = \boldsymbol{v}(t+\tau) - \boldsymbol{v}(t)$, are quasi-Gaussian for time lags τ of order T_L but become increasingly intermittent at higher frequencies [8]. The resulting acceleration statistics exhibit some of the most extreme fluctuations of any known quantity, with accelerations, $\boldsymbol{a}(t)$, up to 80 times its root mean square value, $a_{\rm rms}$, possible [7]. The most natural way to quantify such phenomena is via probability density functions (PDFs) of the Lagrangian velocity increment, $\mathcal{P}(\delta_\tau \boldsymbol{v})$, and acceleration, $\mathcal{P}(\boldsymbol{a})$. The frequency of extreme events is reflected in the size of the tails of the PDFs and thus in the high order moments. These can be analyzed with the aid of Lagrangian velocity structure functions $S_p(\tau) = \langle(\delta_\tau v)^p\rangle$, where $\delta_\tau v$ characterizes the magnitude of a component of the velocity increment. Since the flow here is isotropic the choice of component is immaterial.

It has long been recognized that Eulerian velocity fluctuations in the inertial subrange exhibit anomalous scaling: $\langle(\delta_r u)^p\rangle \equiv \langle(u(x+r) - u(x))^p\rangle \sim r^{\zeta_E(p)}$ [15], where r is the spatial separation. On the basis of simple phenomenological arguments, we may expect the Lagrangian velocity fluctuations to exhibit a power law behavior for time scales within the inertial subrange too. We may therefore assume that $S_p(\tau) \sim \tau^{\zeta_L(p)}$ with $\tau_\eta \ll \tau \ll T_L$. Anomalous scaling is often interpreted as the result of the intermittent nature of the energy cascade. Among the simplest stochastic models able to reproduce both qualitatively and quantitatively such intermittency are those based on the multifractal formalism. This has been successfully used to explain Eulerian statistics such as structure functions [15–17] and velocity gradients [18,19] and Lagrangian statistics such as the acceleration covariance [20] and the velocity statistics [21,22]. The aim of this Letter is to compare predictions of the multifractal formalism for Lagrangian velocity structure functions, the acceleration PDF, and the acceleration variance conditioned on the instantaneous velocity with those obtained from the DNS data. The Lagrangian multifractal predictions are derived from the multifractal formalism in the Eulerian reference frame without any additional free parameters.

In the multifractal formalism the global scale invariance of Kolmogorov's theory (K41) becomes a local scale invariance. Namely, the turbulent flow is assumed to possess a range of scaling exponents $I = (h_{\min}, h_{\max})$. For each $h \in I$ there is a set $S_h \in \mathbb{R}^3$ of fractal dimension $D(h)$ such that, in the limit of small r, $\delta_r u(\boldsymbol{x}) \sim u_0(r/L_0)^{h(\boldsymbol{x})}$ for $\boldsymbol{x} \in S_h$. Here u_0 is the large scale fluctuating velocity and L_0 is the integral length scale. For small values of u_0 we are in the laminar part of the flow for which a multifractal description is not appropriate. From this local scaling law, the scaling properties of the Eulerian structure function can easily be derived by integrating over all possible h [15]: $\langle(\delta_r u)^p\rangle \sim \langle u_0^p\rangle \times \int_I dh(r/L_0)^{hp+3-D(h)}$. The factor $(r/L_0)^{3-D(h)}$ is the probability of being within a distance of order r of the set S_h of dimension $D(h)$. A saddle point approximation in the limit $r \ll L_0$ then gives the scaling exponents

$$\zeta_E(p) = \inf_h[hp + 3 - D(h)]. \qquad (1)$$

Among possible empirical formulas for the scaling exponents, $\zeta_E(p)$, we choose the one of She and Lévêque [23]. Using this it can be shown that

$$D(h) = 1 + p^*(h)(h - \tfrac{1}{9}) + 2(\tfrac{2}{3})^{p^*(h)/3}, \qquad (2)$$

where $p^*(h) = [3/\ln(2/3)]\ln\{(1 - 9h)/[6\ln(2/3)]\}$ is the value of p which minimizes the inverse of (1).

The velocity fluctuations along a particle trajectory may be considered as the superposition of different contributions from eddies of all sizes. In a time lag τ the contributions from eddies smaller than a given scale, r, are uncorrelated and one may then write $\delta_\tau v \sim \delta_r u$. We assume that r and τ are linked by the typical eddy turnover time at the given spatial scale, $\tau_r \sim r/\delta_r u$. Therefore, in the multifractal terminology,

$$\delta_\tau v \sim \delta_r u \qquad \tau \sim \frac{L_0^h}{v_0} r^{1-h}. \qquad (3)$$

The presence of fluctuating eddy turnover times is the only additional complication introduced by the multifractal formalism in the Lagrangian reference frame. Using (3) we can now derive a prediction for the Lagrangian velocity structure function [22]:

$$S_p(\tau) \sim \langle v_0^p\rangle \int_{h\in I} dh\left(\frac{\tau}{T_L}\right)^{[hp+3-D(h)]/(1-h)},$$

where the factor $(\tau/T_L)^{[3-D(h)]/(1-h)}$ is the probability of observing an exponent h in a time lag τ. The exponents $\zeta_L(p)$ then follow from a saddle point approximation in the limit $\tau \ll T_L$:

$$\zeta_L(p) = \inf_h\left(\frac{hp + 3 - D(h)}{1 - h}\right). \qquad (4)$$

In Fig. 1 the results for $S_p(\tau)$ calculated from the DNS are presented. Although the scaling in a log-log plot is reasonable, a more detailed inspection of the logarithmic local slopes, $d\log S_p(\tau)/d\log S_2(\tau)$, displays a deterioration of scaling properties at small times. This is due to the presence of a strong saturation effect for time lags, $\tau \in [\tau_\eta, 10\tau_\eta]$. This may be explained in terms of trapping events inside vortical structures [14], a dynamical effect which may strongly affect scaling properties and which a simple multifractal model cannot capture. For this reason, scaling properties are recovered using only ESS [24] and for large time lags, $\tau > 10\tau_\eta$. In this interval a satisfactory agreement with the multifractal

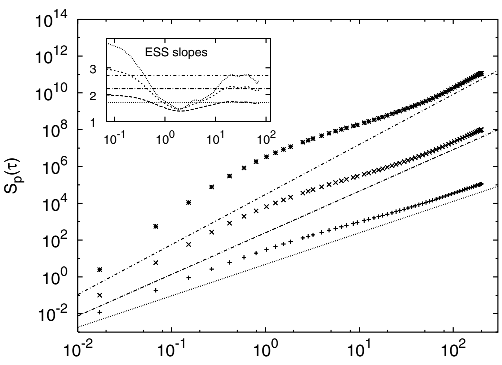

FIG. 1. Extended self-similarity (ESS) plot of the Lagrangian velocity structure function $S_p(\tau)$ versus $S_2(\tau)$, both normalized by the value of the structure function at the Kolmogorov scales. Symbols refer to the DNS data for $p = 8, 6, 4$ from top to bottom. Lines have slopes $\zeta_L(p)/\zeta_L(2)$ given by the multifractal prediction (4). In the inset we show the logarithmic local slopes of the DNS data and the multifractal predictions versus the time lag, τ/τ_η.

prediction (4) is observed, namely, $\zeta_L(4)/\zeta_L(2) = 1.71$; $\zeta_L(6)/\zeta_L(2) = 2.16$; $\zeta_L(8)/\zeta_L(2) = 2.72$.

Similar phenomenological arguments can be used to derive predictions for the acceleration statistics. The acceleration at the smallest scales is defined by

$$a \equiv \frac{\delta_{\tau_\eta} v}{\tau_\eta}. \quad (5)$$

As the Kolmogorov scale, η, fluctuates in the multifractal formalism [15], $\eta(h, v_0) \sim (\nu L_0^h/v_0)^{1/(1+h)}$, so does the Kolmogorov time scale, $\tau_\eta(h, v_0)$. Using (3) and (5) evaluated at η, we get, for a given h and v_0,

$$a(h, v_0) \sim \nu^{(2h-1)/(1+h)} v_0^{3/(1+h)} L_0^{-3h/(1+h)}. \quad (6)$$

The PDF of the acceleration can be derived by integrating (6) over all h and v_0, weighted with their respective probabilities, $[\tau_\eta(h, v_0)/T_L(v_0)]^{[3-D(h)]/(1-h)}$ and $\mathcal{P}(v_0)$. The large scale velocity PDF is reasonably approximated by a Gaussian [15]: $\mathcal{P}(v_0) = \exp(-v_0^2/2\sigma_v^2)/\sqrt{2\pi\sigma_v^2}$, where $\sigma_v^2 = \langle v_0^2 \rangle$. Integration over v_0 gives

$$\mathcal{P}(a) \sim \int_{h \in I} dh a^{[h-5+D(h)]/3} \nu^{[7-2h-2D(h)]/3} L_0^{D(h)+h-3} \sigma_v^{-1} \times \exp\left(-\frac{a^{2(1+h)/3} \nu^{2(1-2h)/3} L_0^{2h}}{2\sigma_v^2}\right). \quad (7)$$

From (7) we can derive the Reynolds number dependence of the acceleration moments [20,25]. For example, in the limit of large R_λ the second order moment is given by $\langle a^2 \rangle \propto R_\lambda^\chi$, where $\chi = \sup_h\{2[D(h) - 4h - 1]/(1 + h)\}$. Thus, we find that $\chi = 1.14$, which differs slightly from the K41 scaling, $\chi^{\mathrm{K41}} = 1$ (see [25–27] for a discussion on departures from K41 scalings in the context of acceleration statistics). In order to compare the DNS data with the multifractal prediction we normalize the acceleration by the rms acceleration, $\sigma_a = \langle a^2 \rangle^{1/2}$. In terms of the dimensionless acceleration, $\tilde{a} = a/\sigma_a$, (7) becomes

$$\mathcal{P}(\tilde{a}) \sim \int_{h \in I} \tilde{a}^{[h-5+D(h)]/3} R_\lambda^{y(h)} \exp\left(-\frac{1}{2} \tilde{a}^{2(1+h)/3} R_\lambda^{z(h)}\right) dh, \quad (8)$$

where $y(h) = \chi[h - 5 + D(h)]/6 + 2[2D(h) + 2h - 7]/3$ and $z(h) = \chi(1 + h)/3 + 4(2h - 1)/3$. We note that (8) may show an unphysical divergence for $a \approx 0$ for many multifractal models of $D(h)$. For example, with $D(h)$ given by (2) we cannot normalize $\mathcal{P}(a)$ for $h < h_c \approx 0.16$. This shortcoming is unimportant for two reasons. First, as already stated, the multifractal formalism cannot be trusted for small velocity and acceleration increments because it is based on arguments valid only to within a constant of order 1. Thus, it is not suited for predicting precise functional forms for the core of the PDF. Second, values of $h \lesssim h_c$ correspond to very intense velocity fluctuations which have never been accurately tested in experiments or by DNS. The precise functional form of $D(h)$ for those values of h is therefore unknown. Thus, we restrict h to be in the range $h_c < h \leq h_{\max}$. For $h_{\max}$ we take the value of h which satisfies $D'(h) = 0$; that is, $h_{\max} \approx 0.38$. Values of $h > h_{\max}$ affect only the peak of the velocity distribution which we have already excluded from our discussion. We also restrict $|\tilde{a}|$ to lie in the range $[\tilde{a}_{\min}, \infty)$ with $\tilde{a}_{\min} = O(1)$.

In Fig. 2 we compare the acceleration PDF computed from the DNS data with the multifractal prediction (8). The large number of Lagrangian particles used in the DNS (see [14] for details) allows us to detect events up to $80\sigma_a$. The accuracy of the statistics was improved by averaging over all directions. Also shown in Fig. 2 is the K41 prediction for the acceleration PDF $\mathcal{P}^{\mathrm{K41}}(\tilde{a}) \sim \tilde{a}^{-5/9} R_\lambda^{-1/2} \exp(-\tilde{a}^{8/9}/2)$ which can be recovered from (8) with $h = 1/3$, $D(h) = 3$, and $\chi^{\mathrm{K41}} = 1$. As is evident from Fig. 2, the multifractal prediction (8) captures the shape of the acceleration PDF much better than the K41 prediction. What is remarkable is that (8) agrees with the DNS data well into the tails of the distribution—from the order of 1 standard deviation, σ_a, up to order $70\sigma_a$. This result is obtained with $D(h)$ given by (2). We emphasize that the only degree of freedom in our formulation of $\mathcal{P}(\tilde{a})$ is the minimum value of the acceleration, $\tilde{a}_{\min}$, here taken to be 1.5. In the inset of Fig. 2 we make a more stringent test of the multifractal prediction (8) by plotting $\tilde{a}^4 \mathcal{P}(\tilde{a})$ and which is seen to agree well with the DNS data.

From (6) it is also possible to derive a prediction for the acceleration moments conditioned on the local—instantaneous—velocity field v_0: $\langle a^n | v_0 \rangle$. Such quantities are important in the construction of Lagrangian stochastic models of turbulent diffusion [2]. For the conditional

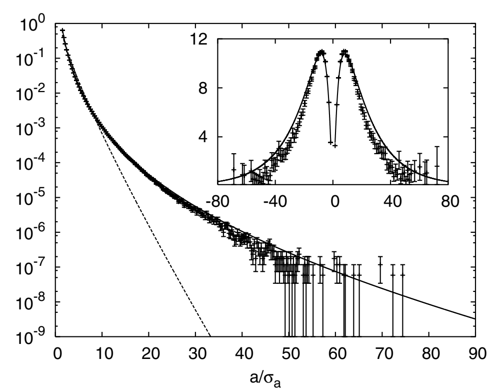

FIG. 2. Log-linear plot of the acceleration PDF. The crosses are the DNS data, the solid line is the multifractal prediction, and the dashed line is the K41 prediction. The DNS statistics were calculated along the trajectories of 2.0×10^6 particles amounting to 1.06×10^{10} events in total. The statistical uncertainty in the PDF was quantified by assuming that fluctuations grow like the square root of the number of events. Inset: $\tilde{a}^4 \mathcal{P}(\tilde{a})$ for the DNS data (crosses) and the multifractal prediction.

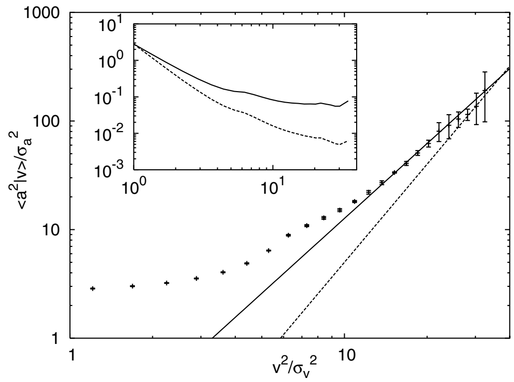

FIG. 3. Log-log plot of the conditional acceleration variance. The crosses are the DNS data, the solid line is the multifractal prediction, and the dashed line is the prediction of [25]. The DNS data are the absolute acceleration conditioned on the absolute velocity. Statistical uncertainty was estimated by dividing the samples into five subensembles. Inset: the conditional acceleration variance scaled by the multifractal prediction, $\langle a^2|v\rangle/v^{4.57}$, and the prediction of [25], $\langle a^2|v\rangle/v^6$, both normalized by σ_a^2.

acceleration variance we get

$$\langle a^2|v_0\rangle \sim \int_{h\in I} dh \nu^{[1+4h-D(h)]/(1+h)} v_0^{[3+D(h)]/(1+h)} \times L_0^{[D(h)-6h-3]/(1+h)}. \quad (9)$$

In the limit $\nu \ll 1$, a saddle point approximation gives $\langle a^2|v_0\rangle \propto \nu^\alpha v_0^{[3+D(\hat{h})]/(1+\hat{h})}$, where $\alpha = \inf_h\{[1+4h-D(h)]/(1+h)\}$ and $\hat{h}$ is the value of h which minimizes the exponent of ν. Thus, we find that $\langle a^2|v_0\rangle \propto v_0^{4.57}$.

In Fig. 3 we plot $\langle a^2|v\rangle$, normalized by the acceleration variance, versus v^2/σ_v^2. The relatively large error that can be seen in the DNS conditional acceleration statistics for large values of v^2/σ_v^2 reflects the rarity of these events. However, in agreement with [25] a clear trend is evident that, for large velocities, the acceleration magnitude depends strongly on the magnitude of the velocity. (The vector acceleration and velocity can easily be shown to be uncorrelated for stationary turbulence [25].) Also shown in Fig. 3 are the multifractal prediction and the prediction of [25] based on a dimensional argument pertaining to the vorticity, namely, that $\langle a^2|v\rangle \propto v^6$. Although statistical noise prevents us from making a convincing claim, the multifractal prediction agrees better with the DNS data.

In conclusion, we have shown that the multifractal formalism predicts a PDF for the unconditional acceleration which is in excellent agreement with the DNS data. Compared with other models [3–5], we have used a very simple phenomenological assumption to derive the form of the PDF. We have assumed only that the Lagrangian velocity increment is related to the Eulerian velocity increment by (3) and that the large scale fluctuating velocity is Gaussian. The only adjustable parameter in our formulation is the value of $\tilde{a}_{\min}$, which does not have a sensitive effect on the results. Lagrangian turbulence also allows a detailed check of the multifractal formalism for spatiotemporal objects as discussed in [28,29]. Work on this will be reported elsewhere.

We thank G. Erbacci and C. Cavazzoni for resource allocation on IBM-SP4 of Cineca (Bologna, Italy). We acknowledge support from the EU under Contracts No. HPRN-CT-2002-00300 and No. HPRN-CT-2000-0162.

[1] R. H. Kraichnan, Phys. Fluids **8**, 575 (1965).
[2] S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, 2000).
[3] C. Beck, Europhys. Lett. **64**, 151 (2003); Phys. Lett. A **287**, 240 (2001).
[4] A. K. Aringazin and M. I. Mazhitov, Phys. Lett. A **313**, 284 (2003).
[5] T. Arimitsu and N. Arimitsu, cond-mat/0301516.
[6] T. Gotoh and R. H. Kraichnan, nlin.CD/0305040.
[7] A. La Porta *et al.*, Nature (London) **409**, 1017 (2001); G. A. Voth *et al.*, J. Fluid Mech. **469**, 121 (2002); N. Mordant *et al.*, physics/0303003.
[8] N. Mordant *et al.*, J. Stat. Phys. **113**, 701 (2003); N. Mordant *et al.*, Phys. Rev. Lett. **89**, 254502 (2002); N. Mordant *et al.*, Phys. Rev. Lett. **87**, 214501 (2001).
[9] S. Ott and J. Mann, J. Fluid Mech. **422**, 207 (2000).
[10] P. K. Yeung, Annu. Rev. Fluid Mech. **34**, 115 (2002). P. K. Yeung, J. Fluid Mech. **427**, 241 (2001); P. Vedula and P. K. Yeung, Phys. Fluids **11**, 1208 (1999).
[11] G. Boffetta and I. M. Sokolov, Phys. Rev. Lett. **88**, 094501 (2002).
[12] T. Ishihara and Y. Kaneda, Phys. Fluids **14**, L69 (2002).
[13] T. Gotoh and D. Fukuyama, Phys. Rev. Lett. **86**, 3775 (2001).
[14] L. Biferale *et al.*, nlin.CD/0402032.
[15] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[16] G. Parisi and U. Frisch, in *Turbulence and Predictability in Geophysical Fluid Dynamics* (North-Holland, New York, 1985), p. 84.
[17] R. Benzi *et al.*, J. Phys. A **17**, 3521 (1984).
[18] R. Benzi *et al.*, Phys. Rev. Lett. **67**, 2299 (1991).
[19] L. Biferale, Phys. Fluids A **5**, 428 (1993).
[20] M. S. Borgas, Philos. Trans. R. Soc. London A **342**, 379 (1993).
[21] L. Chevillard *et al.*, Phys. Rev. Lett. **91**, 214502 (2003).
[22] G. Boffetta *et al.*, Phys. Rev. E **66**, 066307 (2002).
[23] Z.S. She and E. Lévêque, Phys. Rev. Lett. **72**, 336 (1994).
[24] R. Benzi *et al.*, Phys. Rev. E **48**, R29 (1993).
[25] B. L. Sawford *et al.*, Phys. Fluids **15**, 3478 (2003).
[26] V. Yakhot and K. R. Sreenivasan, nlin.CD/0406037.
[27] R. J. Hill, J. Fluid Mech. **452**, 361 (2002).
[28] V. L'vov *et al.*, Phys. Rev. E **55**, 7030 (1997).
[29] L. Biferale *et al.*, Physica (Amsterdam) **127D**, 187 (1999).