Asymmetric Vortex Solitons in Nonlinear Periodic Lattices

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We reveal the existence of asymmetric vortex solitons in ideally symmetric periodic lattices and show how such nonlinear localized structures describing elementary circular flows can be analyzed systematically using the energy-balance relations. We present the examples of rhomboid, rectangular, and triangular vortex solitons on a square lattice and also describe novel coherent states where the populations of clockwise and anticlockwise vortex modes change periodically due to a nonlinearity-induced momentum exchange through the lattice. Asymmetric vortex solitons are expected to exist in different nonlinear lattice systems, including optically induced photonic lattices, nonlinear photonic crystals, and Bose-Einstein condensates in optical lattices.

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Vortices are fundamental objects which appear in many branches of physics [1]. In optics, vortices are associated with phase dislocations (or phase singularities) carried by diffracting optical beams [2], and they share many common properties with the vortices observed in superfluids and Bose-Einstein condensates (BEC) [3]. Vortices may emerge spontaneously or can be generated by various experimental techniques, and they are important objects for fundamental studies also having many practical applications. Nonlinear self-action effects are inherent to many physical systems, and they support the existence of localized vortexlike structures in the form of vortex solitons when the flow around the singularity does not change the density distribution, the property naturally observed for axially symmetric structures in homogeneous media [Fig. 1(a)].

Periodic lattices, such as photonic structures for laser beams or optical lattices for atomic BECs, allow for a strong modification of the wave propagation, which also depends on the energy density. Recently, it was demonstrated experimentally that photonic lattices can support stable off-site (small radius) and on-site (larger radius) vortices on a square lattice [4]. The squarelike profiles of such vortices resemble a homogeneous vortex modulated by the underlying periodic structure [Fig. 1(b)]. Vortex solitons describe elementary circular energy flows, and therefore their properties are intrinsically linked to the wave transport mechanisms in the underlying periodic lattice. In this Letter, we demonstrate that this connection is highly nontrivial due to an interplay between nonlinearity and periodicity. We predict analytically and confirm by numerical simulations that even ideally symmetric periodic structures can support robust asymmetric vortex solitons, and our approach allows for a systematic study of such novel types of singular states. We find that some symmetries are always allowed, whereas other configurations, such as a triangular vortex shown in Fig. 1(c), may exist under certain conditions derived from a balance of the energy flows. Additionally, we predict the existence of fully coherent states in the form of vortices which exhibit a charge flipping, i.e., a periodic reversal of the energy flow. This effect can occur for larger-radius vortices, uncovering a key difference between the on-site and off-site vortex states.

We consider the nonlinear propagation of an optical beam in a two-dimensional periodic lattice [5] described by the dimensionless nonlinear equation:

$$i\frac{\partial \Psi}{\partial z} + D\nabla_\perp \Psi + V(x, y)\Psi - \mathcal{G}(x, y, |\Psi|^2)\Psi = 0, \quad (1)$$

where $\nabla_\perp$ stands for the transverse Laplacian, $\Psi(x, y, z)$ is the complex field amplitude, x, y are the transverse coordinates, z is the propagation coordinate, and D is the diffraction (or dispersion) coefficient. Function V defines a periodic potential of the two-dimensional lattice, and the function $\mathcal{G}$ characterizes a nonlinear response. Similar mathematical models appear for describing the self-action effects in nonlinear photonic crystals [6] and the nonlinear dynamics of atomic BEC in optical lattices [7].

Nonlinearity in Eq. (1) can compensate for the diffraction-induced beam spreading in the transverse dimensions leading to the formation of stationary structures in the form of spatial solitons, $\Psi(x, y, z) = \psi(x, y)e^{i\beta z}$, where $\psi(x, y)$ is the soliton envelope, and β is a nonlinear shift of the propagation constant, the soliton parameter. Periodic lattices can modify dramatically the soliton properties, as was demonstrated for waveguide arrays and dynamically induced periodic photonic lattices [5]. In particular, periodic lattices can stabilize and support propagation of discrete vortex solitons [8–11] which have recently been observed in experiment [4].

In order to analyze the vortexlike structures in a periodic potential, we write the field envelope in the form $\psi(x, y) = |\psi(x, y)| \exp[i\varphi(x, y)]$ and assume that the accumulation of the phase φ around a singular point (at $\psi = 0$) is $2\pi M$, where M is an integer topological charge. We

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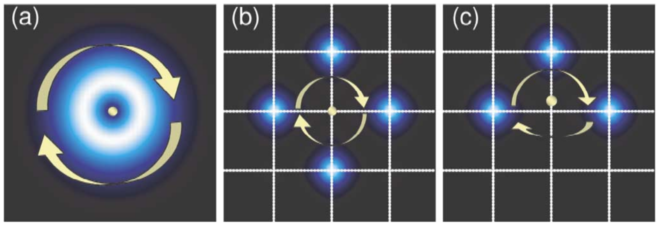

FIG. 1 (color online). Schematic of (a) conventional vortex in a homogeneous medium, (b),(c) symmetric and asymmetric vortex solitons in a periodic lattice. Arrows indicate the direction of the phase change and associated energy flow around the screw dislocation (phase singularity) marked at the vortex center.

consider spatially localized structures in the form of vortexlike bright solitons with the envelopes vanishing at infinity. Such structures can exist when the soliton parameter β is placed inside a gap of the linear Floquet-Bloch spectrum of the periodic structure [6,7].

The profiles of four-lobe symmetric vortex solitons discussed so far in both theory [8–11] and experiment [4] resemble closely a ringlike structure of the soliton clusters [12] in homogeneous media. Then, we look for novel vortex solitons of arbitrary symmetry as a superposition of a number of the fundamental solitons:

$$\Psi(x, y; z) = \sum_{n=0}^{N-1} A_n(z)\psi_s(x - x_n, y - y_n)e^{i\beta z}, \qquad (2)$$

where ψ_s are the profiles of the individual fundamental solitons, $n = 0, \ldots, N - 1$, N is the total number of solitons, (x_n, y_n) are the soliton positions, and A_n is the scaling coefficient defining the phase of the nth soliton and variation of its amplitude due to interaction with other solitons. In contrast to the case of a homogeneous medium [12], the positions of the individual solitons are fixed by the lattice potential, provided the lattice potential is sufficiently strong. In order to determine the soliton amplitudes, we present Eq. (1) in the Hamiltonian form $id\Psi/dz = \delta\mathcal{H}(\Psi, \Psi^*)/\delta\Psi^*$ and derive, after substituting Eq. (2) into the full Hamiltonian, the reduced Hamiltonian $H_s(A_n, A_n^*)$. The resulting amplitude equations can be written in the form

$$i\frac{dA_n}{dz} = \frac{\delta H_s}{\delta A_n^*} = -\sum_{m=0}^{N-1} c_{nm}A_m - G(|A_n|^2)A_n - F_n, \qquad (3)$$

where $c_{nm} \equiv c_{mn} = \int\int \mathcal{G}[x, y, |\psi_s(x - x_n, y - y_n)|^2] \times \psi_s(x - x_n, y - y_n)\psi_s^*(x - x_m, y - y_m)dxdy / \int\int \psi_s(x, y) \times \psi_s^*(x, y)dxdy$ are the coupling coefficients, G is the effective local nonlinearity, and F_n defines the nonlinear coupling terms such as $\sim A_{m_1}A_{m_2}^*A_n$. We note that the approximation (2) is valid when $c_{n\neq m}, G, F/A_n \sim \varepsilon \ll 1$. As was demonstrated earlier [13,14], under such conditions the amplitudes A_n are only slightly perturbed, and we can seek stationary solutions of Eq. (3) corresponding to vortex solitons by means of the perturbation theory: $A_n = [1 + O(\varepsilon)]\exp[i\varphi_n + iO(\varepsilon)]$. In Eq. (3), the nonlinear coupling terms are proportional to the forth-order overlap integrals and, therefore, $F_n \ll c_{nm}A_m$. Then, in the first order we obtain a general constraint on the soliton phases φ_n:

$$\sum_{m=0}^{N-1} c_{nm}\sin(\varphi_m - \varphi_n) = 0. \qquad (4)$$

In the sum (4), each term defines the energy flow between the solitons with numbers n and m so that the Eq. (4) represents a condition for a balance of the energy flows which is required for stable propagation of a soliton cluster and the vortex-soliton formation. These conditions are satisfied trivially when all the solitons are in- or out-of-phase. We note that Eq. (3) with $F_n \equiv 0$ has the form of a discrete self-trapping equation, which appears in different physical contexts [15,16]. However, nontrivial solutions of Eq. (4) corresponding to the vortexlike soliton clusters have been analyzed only for symmetric configurations, and even then some important solutions have been overlooked, as we demonstrate below. Moreover, we show that the existence domains of asymmetric vortexlike solutions are highly nontrivial due to specific properties of the coupling coefficients calculated for realistic periodic lattices.

In order to provide a direct link to the recent experimental results [4], first we apply our general analytical approach to describe the vortex solitons in two-dimensional optically induced lattices created in a photorefractive crystal. Then, the diffraction coefficient in Eq. (1) is defined as $D = z_0\lambda/(4\pi n_0 x_0^2)$, where x_0 and z_0 are the characteristic length scales in the transverse and longitudinal spatial dimensions, respectively, n_0 is the average medium refractive index, and λ is the vacuum wavelength. The lattice potential and nonlinear beam self-action effect are both due to the photorefractive screening nonlinearity:

$$\mathcal{G} - V = \gamma\left\{I_b + I_0\sin^2\left(\frac{\pi x}{d}\right)\sin^2\left(\frac{\pi y}{d}\right) + |\Psi|^2\right\}^{-1}, \qquad (5)$$

where γ is proportional to the external bias field, I_b is the dark irradiance, and I_0 is the intensity of interfering beams that induce a square lattice with the period d (see details in Refs. [4,17]).

We use nonlinear localized (soliton) solutions obtained by using the ansatz (2) as initial conditions for a numerical relaxation algorithm applied to the model (1) and (5) with the normalized coefficients $D = 0.5$, $I_b = 1$, and $d = 2$. We start by considering a four-soliton vortex on a square lattice [Fig. 2(a)]. The previously reported solutions [Figs. 2(b) and 2(c)] have both the reflection and 90° rotational symmetries, similar to the underlying lattice. Accordingly, $c_{01} = c_{12} = c_{23} = c_{30}$ and $c_{02} = c_{13}$, and Eq. (4) has a nontrivial-phase solution with $\varphi_n = n\pi/2$ corresponding to a charge-one symmetric vortex. However, our analytical approach allows us to predict more general, reduced-symmetry vortex solitons when the 90° rotational symmetry is removed. In particular, we find a

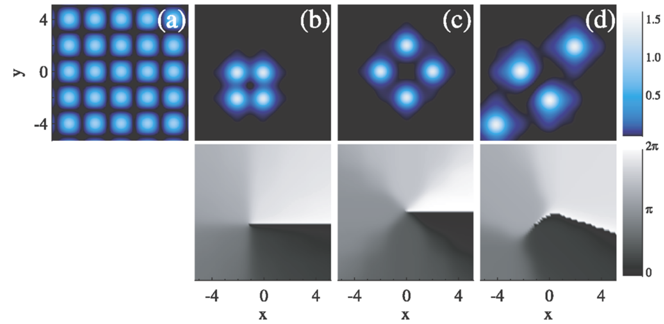

FIG. 2 (color online). Examples of the vortex-type soliton structures with various symmetries in a square lattice potential shown in (a). (b),(c) Off-site and on-site symmetric square vortex solitons ($\beta = 5.5$, $\gamma = 8$, $I_0 = 1$); (d) rhomboid configuration ($\beta = 3.5$, $\gamma = 5$, $I_0 = 1$) with a topological charge $+1$. Shown are the intensity profiles (top) and phase structures (bottom).

rhomboid configuration that has a vortex charge with $\varphi_n = n\pi/2$ [Fig. 2(d)]. We found that rhomboid vortices are remarkably robust, suggesting that their generation in experiment can be possible by using elliptically shaped singular beams.

Most remarkably, for both square and rhomboid configurations, we find that the balance Eq. (4) admits more general exact solutions, which were overlooked in the earlier studies [15] of Eq. (3), namely, $\varphi_0 = \varphi_2 - \pi$ and $\varphi_1 = \varphi_3 - \pi$, where the phase difference $\varphi_1 - \varphi_0$ is arbitrary. These novel solutions describe a family of vortex solitons having the same intensity profile but different phase structure. Because of such a degeneracy, a small change in the amplitude of two opposing solitons can initiate a slow variation of the free phase, $\varphi_1 - \varphi_0 \simeq \kappa z$; this regime corresponds to a periodic flipping of the vortex charge. Although the general Eq. (3) is satisfied only when $\varphi_1 - \varphi_0 = 0, \pi/2, \ldots$, the charge-flipping effect can be induced by a finite perturbation when the nonlinear coupling terms (F_n) are small. Indeed, we find that a closely packed vortexlike state shown in Fig. 2(b) is resistant to the charge flipping effect, similar to vortices in single-well potentials [18], whereas the vortex shown in Fig. 2(c) can exhibit the charge flipping after increasing the amplitudes of two opposing solitons by 7%, as shown in Fig. 3. The bottom plot clearly illustrates that the solitons strongly exchange energy; however, the flows always remain balanced. These are novel coherent states, where the populations of clockwise and anticlockwise rotational modes change periodically due to nonlinearity-induced momentum exchange through the lattice.

We now consider another example: vortices with rectangular arrangement of the fundamental solitons. In this case, $c_{01} = c_{23}$, $c_{03} = c_{12}$, and $c_{02} = c_{13}$. With no loss of generality, we assign the soliton numbers such that $c_{01} \leq c_{03}$, and then nontrivial solutions of Eq. (4) are $\varphi_1 - \varphi_0 = \cos^{-1}(-c_{02}/c_{03})$, $\varphi_2 - \varphi_0 = \cos^{-1}(-c_{01}/c_{03})$, and $\varphi_3 + \varphi_0 = \varphi_1 + \varphi_2$. Since an inverse cosine function has two branches, there exist four solutions corresponding to

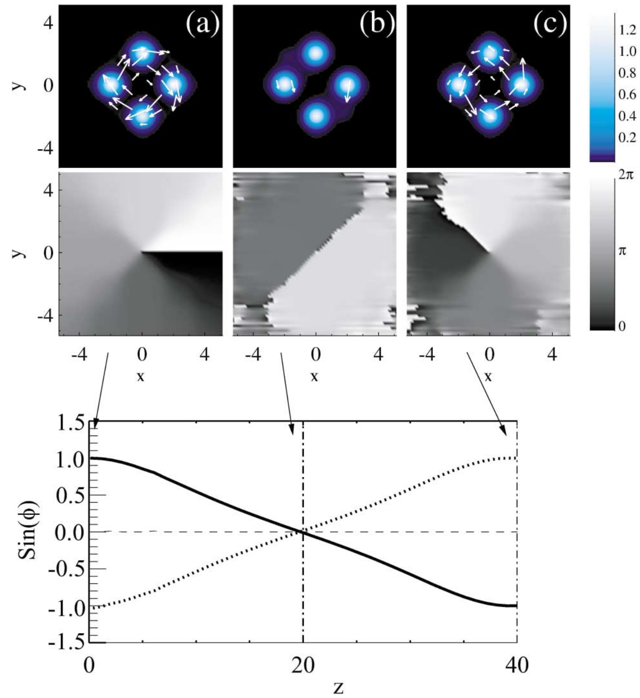

FIG. 3 (color online). Charge flipping effect for a square vortex of Fig. 2(c) induced by a 7% increase of the amplitudes of two opposite solitons. Top: Snapshots at increasing propagation distances showing the unchanged intensity profile. Arrows show the energy flow. Bottom: energy flows between the solitons characterized by sinusoidal functions of the phase differences between the opposite (dashed line) and neighboring (solid and dotted lines) solitons according to Eq. (4).

two pairs of positively and negatively charged vortices with different positions of singularity. We find that the singularity is always shifted away from the center of rectangle along its longer dimension, as illustrated in Figs. 4(b) and 4(c). This happens due to a highly asymmetric phase structure of the vortex, which in turn can

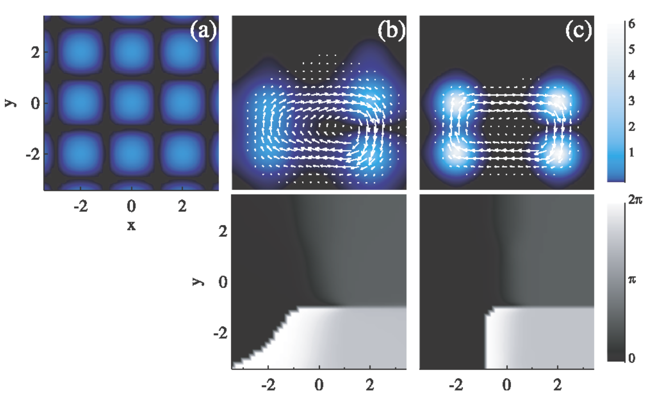

FIG. 4 (color online). (a) Lattice potential; (b),(c) examples of asymmetric vortex solitons with rectangle configurations for (b) $\beta = 5.5$, $\gamma = 8$, $I_0 = 1$ and (c) $\beta = 6$, $\gamma = 16.09$, $I_0 = 3$. Top: intensity profiles with arrows indicating the energy flow. Bottom: phase structures.

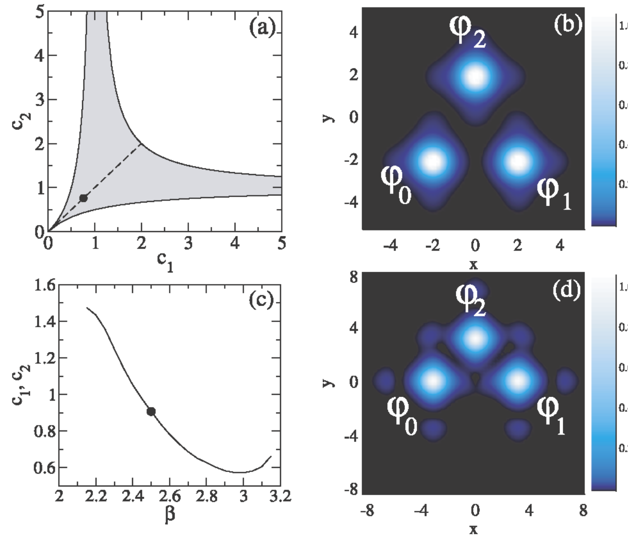

FIG. 5 (color online). (a) Existence region in the plane of normalized coupling coefficients for the three-soliton vortex; (b) example of a vortex soliton with the phase configuration $\varphi_0 = 0$, $\varphi_0 \simeq 0.981\pi$, $\varphi_2 \simeq -0.5097\pi$, supported by a self-focusing photorefractive screening nonlinearity ($\beta = 5.5$, $\gamma = 8$, $I_0 = 1$); (c) dependence of the coupling coefficients on the soliton propagation constant β inside the band gap for a configuration show in (d); (d) example of a vortex soliton with the phase configuration $\varphi_0 = 0$, $\varphi_1 \simeq 0.624\pi$, $\varphi_2 \simeq -0.625\pi$, in repulsive atomic condensate ($\beta = 2.5$, $\gamma = 1$, $V_0 = 2.5$, $d = \pi/2$) corresponding to marked points in plots (a),(c).

lead to deformations of the vortex intensity profile, resulting in a trapezoidlike shape shown in Fig. 4(b).

Finally, we mention another remarkable example of asymmetric vortex solitons in the form of a triangular structure. We find that Eq. (4) possesses nontrivial solutions for $N = 3$ only when the normalized coupling coefficients, $c_1 = c_{12}/c_{01}$ and $c_2 = c_{02}/c_{01}$, satisfy the following conditions, $(1 + |c_1|^{-1})^{-1} < |c_2| < |1 - |c_1|^{-1}|^{-1}$, and we show the existence domain in Fig. 5(a). It follows that highly asymmetric triangular vortices are not allowed. On the other hand, isosceles configurations are always possible if the soliton base is sufficiently narrow, so that condition $c_1 = c_2 < 2$ is satisfied. We show an example of such vortex soliton in Fig. 5(b).

In order to underline the generality and broad applicability of our results, we also search for asymmetric vortex solitons in repulsive atomic Bose-Einstein condensates loaded onto a two-dimensional optical lattice. In the mean-field approximation, this system is described by the Gross-Pitaevskii Eq. (1) with

$$V = V_0[\sin(\pi x/d) + \sin(\pi y/d)]; \qquad \mathcal{G} = \gamma|\Psi|^2, \quad (6)$$

where $\gamma > 0$, and V_0 is the depth of an optical-lattice potential. Repulsive nonlinearity can lead to wave localization in the form of gap solitons [7], which exhibit effective long-range interaction. This allows, in particular, the existence of triangular vortices with a wider base [Fig. 5(d)] compared to the conventional solitons in the self-focusing regime [cf. Figure 5(b)]. We find that such triangular gap vortices exist throughout the whole band gap since $c_1 = c_2 < 2$ [Fig. 5(c)], except for the immediate vicinity of the spectrum band edges where our approximation (2) is not applicable.

In conclusion, we have revealed that periodic lattices can support different types of robust asymmetric vortex-like nonlinear localized structures. Such vortices resemble the soliton clusters trapped by the lattice, and they are associated with a nontrivial power flow. We have presented the examples of novel vortex solitons on a square lattice, and other solutions can be obtained and analyzed using the general energy-balance relations. We believe our findings will initiate the experimental efforts to observe such vortices in optically induced photonic structures, Bose-Einstein condensates in optical lattices, photonic crystals, and photonic crystal fibers.

[1] L. M. Pismen, *Vortices in Nonlinear Fields* (Clarendon Press, Oxford, 1999), p. 290.
[2] M. S. Soskin and M. V. Vasnetsov, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 2001), Vol. 42, p. 219.
[3] K. W. Madison *et al.*, Phys. Rev. Lett. **84**, 806 (2000); C. Raman *et al.*, Phys. Rev. Lett. **87**, 210402 (2001).
[4] D. Neshev *et al.*, Phys. Rev. Lett. **92**, 123903 (2004); J. W. Fleischer *et al.*, Phys. Rev. Lett. **92**, 123904 (2004).
[5] D. N. Christodoulides *et al.*, Nature (London) **424**, 817 (2003).
[6] S. F. Mingaleev and Yu. S. Kivshar, Phys. Rev. Lett. **86**, 5474 (2001).
[7] E. A. Ostrovskaya and Yu. S. Kivshar, Phys. Rev. Lett. **90**, 160407 (2003); Opt Express **12**, 19 (2004).
[8] B. A. Malomed and P. G. Kevrekidis, Phys. Rev. E **64**, 026601 (2001).
[9] P. G. Kevrekidis *et al.*, Phys. Rev. E **66**, 016609 (2002).
[10] M. Johansson *et al.*, Physica (Amsterdam) **119D**, 115 (1998).
[11] J. Yang and Z. H. Musslimani, Opt. Lett. **28**, 2094 (2003); J. Yang, New J. Phys. **6**, 47 (2004).
[12] A. S. Desyatnikov and Yu. S. Kivshar, Phys. Rev. Lett. **88**, 053901 (2002).
[13] R. S. MacKay and S. Aubry, Nonlinearity **7**, 1623 (1994).
[14] S. Aubry, Physica (Amsterdam) **103D**, 201 (1997).
[15] J. C. Eilbeck *et al.*, Physica (Amsterdam) **16D**, 318 (1985).
[16] J. C. Eilbeck and M. Johansson, nlin.PS/0211049, and references therein.
[17] N. K. Efremidis *et al.*, Phys. Rev. Lett. **91**, 213906 (2003).
[18] J. J. Garcia Ripoll *et al.*, Phys. Rev. Lett. **87**, 140403 (2001).